Lecture 1: Intro, plane waves

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office hours : Tuesday 4pm - 6pm
Midterm : Oct 13 10:30am - 11:50am
Final : Dec 16 10:30am - 12:30 pm

Notation
Scalar : k
Vector : $\vec{k}$
unit vectors : $\hat{I}, \hat{J}, \hat{k}$ in x, y, z direction
Complex number : $\underline{k}$

$$c = \frac{\lambda}{T} = \frac{\lambda}{1/f} = \frac{2\pi\lambda}{2\pi/f} = \frac{k}{\omega}$$

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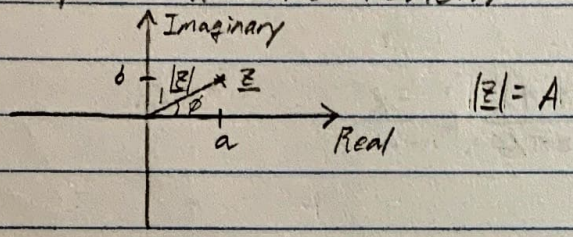
Lecture 2 : Plane waves, complex numbers review

* Temperal angular frequency: ω
* Temperal frequency : $f = \frac{c}{\lambda}$
* Temperal period : $T = \frac{1}{f} = \frac{2\pi}{\omega}$
* Wave vector : $\vec{k}$ [amplitude : $|\vec{k}| = k = \frac{2\pi}{\lambda}$ (λ : wave length), direction of propagation]

$$c = f\lambda \Rightarrow k = \frac{\omega}{c}$$

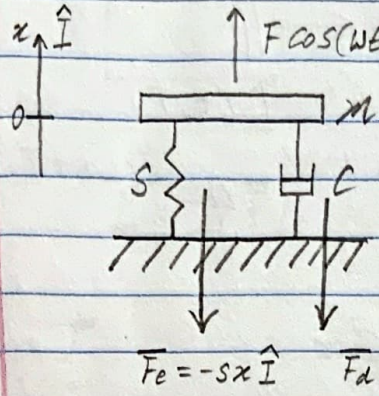
(Complex numbers review)



$|z| = A$

	Real-Imaginary representation	Polar representation
①	$z = a + jb$	$z = A\cos\phi + jA\sin\phi$ (Euler's formular) $= Ae^{j\phi}$ (ϕ : phase, A : amplitude)
②	$\lvert z\rvert = \sqrt{a^2+b^2} = A$ $\phi = \begin{cases} \text{atan}\frac{b}{a} & (a>0) \\ \pi + \text{atan}\frac{b}{a} & (a<0) \end{cases}$	Real : $A\cos\phi$ Im : $A\sin\phi$
③	Addition $z_1 = a_1 + jb_1$, $z_2 = a_2 + jb_2$ $z_1 + z_2 = (a_1+a_2) + j(b_1+b_2)$	$z_1 = A_1 e^{j\phi_1}$, $z_2 = A_2 e^{j\phi_2}$ $z_1 + z_2 = A_1\cos\phi_1 + A_2\cos\phi_2$ $+ j(A_1\sin\phi_1 + A_2\sin\phi_2)$
④	Multiplication $z_1 z_2 = a_1a_2 - b_1b_2$ $+ j(a_1b_2 + a_2b_1)$	$z_1 z_2 = A_1A_2 e^{j(\phi_1+\phi_2)}$ $(\lvert z_1z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert,\ \angle z_1z_2 = \angle z_1 + \angle z_2)$
⑤	Inverse $\frac{1}{z} = \frac{1}{a+jb} = \frac{a-jb}{a^2+b^2}$	$\frac{1}{z} = \frac{1}{Ae^{j\phi}} = \frac{1}{A}e^{-j\phi}$
⑥	Conjugate $z^* = a - jb$	$z^* = Ae^{-j\phi}$

Why only take the real part?



$$F\cos(\omega t) - sx - c\dot{x} = m\ddot{x}$$
$$\Rightarrow m\ddot{x} + c\dot{x} + sx = F\cos(\omega t)$$

Solution: $x(t) = x_t(t) + x_s(t)$

$x_t(t)$: temporal solution

$x_s(t)$: steady state solution

ME240: $x_s(t) = A\cos(\omega t + \phi)$

$$A = \frac{F}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}, \quad \phi = -\operatorname{atan}\left(\frac{c\omega}{s - m\omega^2}\right)$$

ME424: $m\ddot{x} + c\dot{x} + sx = F\cos(\omega t)$

$$= Re\{F\cos(\omega t) + jF\sin(\omega t)\} = Re\{Fe^{j\omega t}\}$$

(Idea: For linear differential equations, the real and imaginary parts are decoupled. The real part of $\underline{x}$ is the solution to the original equation.)

$$\underline{x}_s = \underline{A}e^{j\omega t}$$
$$\dot{\underline{x}}_s = j\omega\underline{A}e^{j\omega t}$$
$$\ddot{\underline{x}}_s = -\omega^2\underline{A}e^{j\omega t}$$

$$\Rightarrow -m\omega^2\underline{A}e^{j\omega t} + jc\omega\underline{A}e^{j\omega t} + s\underline{A}e^{j\omega t} = Fe^{j\omega t}$$

$$\Rightarrow \underline{A} = \frac{F}{s - m\omega^2 + jc\omega}$$

$$A = \frac{F}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

$$\phi = -\operatorname{atan}\left(\frac{c\omega}{s - m\omega^2}\right)$$

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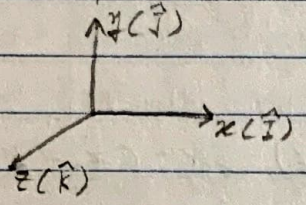
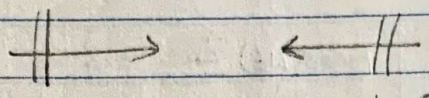
Lecture 3 : Plane waves

Example) Standing Wave





$\underline{P}_1(t,\bar{r}) = \underline{A} e^{j(\omega t - \bar{k}\cdot\bar{r})}$

$\underline{P}_2(t,\bar{r}) = \underline{A} e^{j(\omega t + \bar{k}\cdot\bar{r})}$

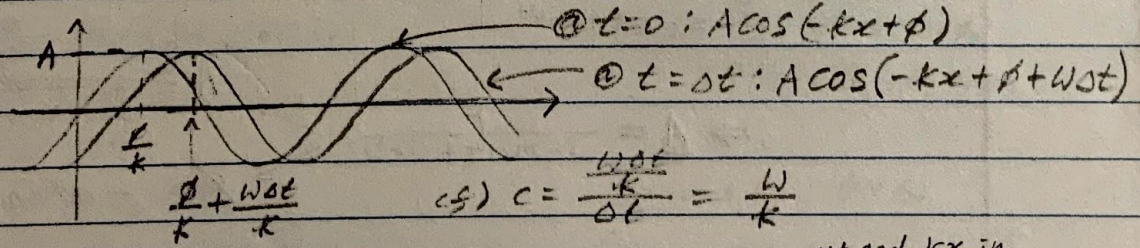
$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$\bar{k} = \frac{\omega}{c}\hat{i}$
$\Rightarrow \bar{k}\cdot\bar{r} = \frac{\omega}{c}x$

$$\begin{aligned}
\underline{P}(t,\bar{r}) &= \underline{A} e^{j(\omega t - \bar{k}\cdot\bar{r})} + \underline{A} e^{j(\omega t + \bar{k}\cdot\bar{r})} \\
&= \underline{A} e^{j\omega t}\left(e^{-j\frac{\omega}{c}x} + e^{j\frac{\omega}{c}x}\right) \\
&= \underline{A} e^{j\omega t}\; 2\cos\left(\tfrac{\omega}{c}x\right) \\
&= 2\underline{A}\cos(kx)\, e^{j\omega t} \\
&= 2A\cos(kx)\, e^{j(\omega t+\phi)} \quad \text{(Standing wave)}
\end{aligned}$$

$Re\{\underline{P}\} = 2A\cos(kx)\cos(\omega t+\phi)$
↳ What we would measure

Example) Traveling plane wave

$\underline{P}_1(t,x) = A e^{j(\omega t - kx + \phi)}$
$P_1(t,x) = Re\{\underline{P}_1\} = A\cos(\omega t - kx + \phi)$

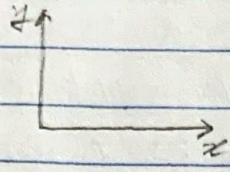


c) $c = \frac{\frac{\omega\Delta t}{k}}{\Delta t} = \frac{\omega}{k}$

Standing wave: $\cos(kx)\cos(\omega t+\phi)$ ← ωt and kx in different cosine
Traveling wave: $\cos(\omega t - kx + \phi)$ → ωt and kx in same cosine

$\frac{3}{2}$ $\frac{1}{2}$ $\frac{2}{\sqrt{5}}$ $\frac{2}{\sqrt{13}}$ $\frac{1}{\sqrt{13}}$

Example) Standing Wave from different amplitudes

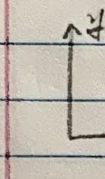


$\underline{P}_1 = \underline{A} e^{j(\omega t - kx)}$ $\quad$ $\underline{P}_2 = \frac{\underline{A}}{2} e^{j(\omega t + kx)}$

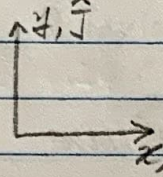
$$\underline{P} = \underline{P}_1 + \underline{P}_2 = \underline{A} e^{j(\omega t - kx)} + \frac{\underline{A}}{2} e^{j(\omega t + kx)}$$
$$= \underline{A} e^{j\omega t} \left(e^{-jkx} + \frac{1}{2} e^{jkx} \right)$$
$$= \underline{A} e^{j\omega t} \left(e^{-jkx} + \frac{1}{2} e^{jkx} + \frac{1}{2} e^{jkx} - \frac{1}{2} e^{jkx} \right)$$
$$= \underline{A} e^{j\omega t} \left(e^{-jkx} + e^{jkx} - \frac{1}{2} e^{jkx} \right)$$
$$= \underline{A} e^{j\omega t} \left(2\cos(kx) - \frac{1}{2} e^{jkx} \right)$$
$$= 2\underline{A} e^{j\omega t} \cos(kx) - \frac{\underline{A}}{2} e^{j(\omega t + kx)}$$

$$Re(\underline{P}) = 2A \cos(\omega t + \phi) \cos(kx) - \frac{A}{2} \cos(\omega t + kx + \phi)$$

$\Rightarrow$ ① $2A$ standing wave, $\frac{A}{2}$ traveling wave ($-x$ direct.)
② A standing wave, $\frac{A}{2}$ traveling wave ($+x$ direct.)



Example) Two waves same direction



$\underline{P}_1 = \underline{A} e^{j(\omega t - kx)}$

$\underline{P}_2 = \underline{B} e^{j(\omega t - kx)}$

$$\underline{P} = \underline{P}_1 + \underline{P}_2 = \underline{A} e^{j(\omega t - kx)} + \underline{B} e^{j(\omega t - kx)} = (\underline{A} + \underline{B}) e^{j(\omega t - kx)}$$

↳ Another plane wave

Approach 2:
$$= \underline{A} e^{j\omega t} \left(\frac{1}{2} e^{-jkx} + \frac{1}{2} e^{jkx} + \frac{1}{2} e^{-jkx} \right)$$
$$= \underline{A} e^{j\omega t} \left(\cos(kx) + \frac{1}{2} e^{-jkx} \right)$$
$$= \underline{A} e^{j\omega t} \cos(kx) + \frac{\underline{A}}{2} e^{j(\omega t - kx)}$$
$$Re(\underline{P}) = A \cos(\omega t + \phi) \cos(kx) + \frac{A}{2} \cos(\omega t - kx + \phi)$$

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Lecture 4 : Comsol Multiphysics

20 μPa is about the smallest sound people can hear

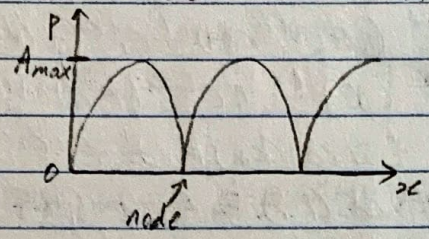
▨ : Domain condition

▭ : Boundary condition

Mesh size : usually $\frac{\lambda}{}$

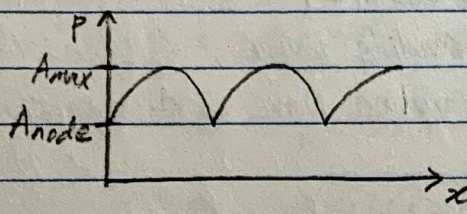
* Standing wave ratio

: Ratio between traveling plane wave and standing wave

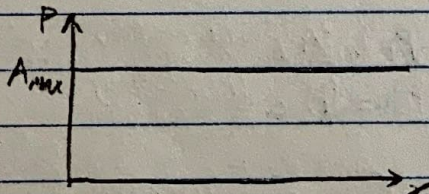


Pure standing wave

$\frac{A_{max}}{A_{node}} = \infty$



Standing wave
+ Traveling wave



Pure traveling wave

$\frac{A_{max}}{A_{node}} = 1$

Lecture 5 : Acoustic wave equation

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Notation

$P(t,\bar{r})$: instantaneous pressure

P_0 : mean pressure, equilibrium pressure (sea level : 100 kPa)

$p(t,\bar{r})$: acoustic pressure $\triangleq P(t,\bar{r}) - P_0$

$\rho(t,\bar{r})$: instantaneous mass density

ρ_0 : mean (equilibrium) mass density

$s = \frac{\rho(t,\bar{r}) - \rho_0}{\rho_0}$: condensation

$\bar{v}(t,\bar{r})$: instantaneous particle velocity

〈Equation of state〉

$P = \rho r T_k$ (r: gas constant $= \frac{\text{universal gas const. } (R_w)}{\text{molar mass } (M_w)}$, T_k : temp. in [K])

Ⓐ Isothermal assumption (T_k is constant)

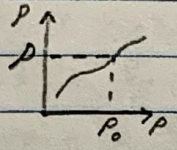
$\frac{P}{P_0} = \frac{\rho}{\rho_0}$

Ⓑ Adiabatic assumption (heat doesn't leave the system)

$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$, γ : ratio of specific heat in the medium

↳ Works well for gases, but not so well for solids and liquids.

Ⓒ Taylor approximation of experimental measurements



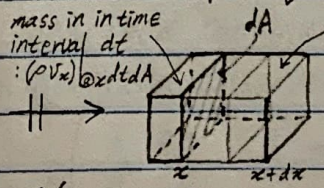
$$P = P_0 + \frac{\partial P}{\partial \rho}\Big|_{@\rho_0}(\rho-\rho_0) + \frac{1}{2}\frac{\partial^2 P}{\partial \rho^2}(\rho-\rho_0)^2 + \cdots$$

$$\approx P_0 + \frac{\partial P}{\partial \rho}\Big|_{@\rho_0}(\rho-\rho_0)$$

$$\Rightarrow P - P_0 = \frac{\partial P}{\partial \rho}\Big|_{@\rho_0}\underbrace{\frac{\rho-\rho_0}{\rho_0}}_{s(t,\bar{r})\ \text{(condensation)}}\rho_0 \Rightarrow P(t,\bar{r}) = \underbrace{\rho_0 \frac{\partial P}{\partial \rho}\Big|_{@\rho_0}}_{B\ \text{(bulk modulus)}} s(t,\bar{r})$$

$\therefore p(t,\bar{r}) = B\, s(t,\bar{r})$ (Equation of state)

〈Continuity equation〉



Mass change in the time interval dt

$: \rho(t+dt)\,dx\,dA - \rho(t)\,dx\,dA$

Change in mass in the interval = mass in − mass out

$$\rho(t+dt) - \rho(t)\,dx\,dA = (\rho v_x)|_{@x} dt\,dA - (\rho v_x)|_{@x+dx}\,dt\,dA$$

$$\frac{\rho(t+dt)-\rho(t)}{dt} = \frac{\partial \rho}{\partial t} = \frac{(\rho v_x)|_{@x} - (\rho v_x)|_{@x+dx}}{dx} = -\frac{\partial(\rho v_x)}{\partial x}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} = 0 \xrightarrow{\text{same for all direct.}} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \bar{v}) = 0$$

$\therefore \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \bar{v}) = 0$ (conservation of mass / continuity eq.)

Del: $\nabla = \frac{\partial}{\partial x}\hat{I} + \frac{\partial}{\partial y}\hat{J} + \frac{\partial}{\partial z}\hat{K}$

Gradient: $\nabla f = \frac{\partial f}{\partial x}\hat{I} + \frac{\partial f}{\partial y}\hat{J} + \frac{\partial f}{\partial z}\hat{K}$

(f: scalar field, ∇f: vector pointing in the direction of greatest increase)

Divergence: $\nabla \cdot \bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

($\bar{a}$: vector field, $\nabla \cdot \bar{a}$: scalar indicating the vector field's tendency to converge or diverge)

Curl: $\nabla \times \bar{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{I} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{J} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{K}$

($\bar{a}$: vector field, $\nabla \times \bar{a}$: vector indicating the torque)

Laplacian: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Lecture 6 : Acoustic wave equation

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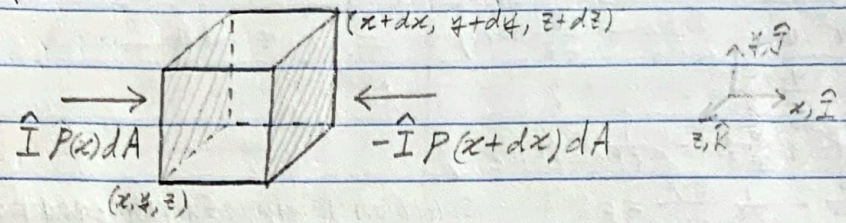
⟨Conservation of momentum⟩



$$\hat{I}: P(x)dA - P(x+dx)\,dA = \frac{\partial}{\partial t}\left[(\rho\,dx\,dA)\,v_x\right]$$

$$-\frac{\partial P}{\partial x}dx\,dA = dx\,dA\,\frac{\partial}{\partial t}(\rho v_x)$$

$$\Rightarrow \frac{\partial P}{\partial x} + \frac{\partial}{\partial t}(\rho v_x) = 0 \quad (\text{same for } \hat{J} \text{ and } \hat{K})$$

$$\hat{I}\frac{\partial P}{\partial x} + \hat{J}\frac{\partial P}{\partial y} + \hat{K}\frac{\partial P}{\partial z} + \frac{\partial}{\partial t}(\hat{I}\rho v_x + \hat{J}\rho v_y + \hat{K}\rho v_z) = 0$$

$$\Rightarrow \nabla P + \frac{\partial}{\partial t}(\rho\bar{v}) = 0 \quad \text{Conservation of momentum}$$

⟨Linearized continuity equation⟩

$$\frac{\partial}{\partial t}(\rho_0 s + \rho_0) + \nabla\cdot(\rho_0 s\bar{v} + \rho_0\bar{v}) = 0$$

$$\rho_0\frac{\partial s}{\partial t} + \frac{\partial \rho_0}{\partial t}^{\to 0} + \rho_0\nabla\cdot(s\bar{v})^{\text{small}} + \rho_0\nabla\cdot\bar{v} = 0$$

(small amplitude assumption: $s, |\bar{v}|, p$ are very small)

$$\Rightarrow \frac{\partial s}{\partial t} + \nabla\cdot\bar{v} = 0 \quad \text{(Linearized continuity eq.)}$$

⟨Linearized conservation of momentum⟩

$$\nabla(\rho_0 + p) + \frac{\partial}{\partial t}(\rho_0 s\bar{v} + \rho_0\bar{v}) = 0$$

$$\nabla\rho_0^{\to 0} + \nabla p + \rho_0\frac{\partial}{\partial t}(s\bar{v})^{\text{small}} + \rho_0\frac{\partial\bar{v}}{\partial t} = 0$$

$$\Rightarrow \nabla p + \rho_0\frac{\partial\bar{v}}{\partial t} = 0 \quad \text{(Linearized conservation of momentum)}$$

Equation of state : $p = Bs$

Linearized continuity eq. : $\frac{\partial s}{\partial t} + \nabla\cdot\bar{v} = 0$ $\Big]\rightarrow \frac{1}{B}\frac{\partial p}{\partial t} + \nabla\cdot\bar{v} = 0$

Linearized conservation of momentum : $\nabla p + \rho_0\frac{\partial\bar{v}}{\partial t} = 0$ $\longrightarrow \frac{\nabla p}{\rho_0}$

$$\frac{d}{dt}\left(\frac{1}{B}\frac{\partial p}{\partial t} + \nabla\cdot\bar{v}\right) = \frac{1}{B}\frac{\partial^2 p}{\partial t^2} + \nabla\cdot\frac{\partial\bar{v}}{\partial t} = \frac{1}{B}\frac{\partial^2 p}{\partial t^2} + \nabla\cdot\left(-\frac{\nabla p}{\rho_0}\right) = 0$$

$$\Rightarrow \nabla\cdot\nabla p - \frac{\rho_0}{B}\frac{\partial^2 p}{\partial t^2} = 0 \quad \left(\nabla\cdot\nabla = \nabla^2 = \underbrace{\Delta}_{\text{Laplacian}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\therefore \nabla^2 p - \frac{1}{B/\rho_0}\frac{\partial^2 p}{\partial t^2} = 0 \quad \text{(Wave equation)}$$

↳ Expression of p in terms of t and x

Notes

① 1D Sound wave $\left(\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0\right)$

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$\frac{\partial^2 p}{\partial x^2} + \cancel{\frac{\partial^2 p}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 p}{\partial z^2}}^{0} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

solution is any function of $t \pm \frac{x}{c}$

$$p(t,x) = p\left(t \pm \frac{x}{c}\right)$$

└ moving function ∓ direction

② Fourier Transform

$$p\left(t \pm \frac{x}{c}\right) = \int \underbrace{A(\omega)}_{\text{└ function of frequency}} e^{i\left(\omega t \pm \frac{\omega}{c}x\right)} d\omega$$

③ Sound wave containing one frequency is called harmonic wave

$$\rightarrow p(t,\bar{r}) = \tilde{p}(\bar{r})\, e^{i\omega t} \quad (p: \text{pressure in terms of } t,\bar{r} \;/\; \tilde{p}: \text{pressure in terms of } \bar{r})$$

$$\Rightarrow \nabla^2\left[\tilde{p}(\bar{r})\, e^{i\omega t}\right] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\tilde{p}(\bar{r})\, e^{i\omega t}\right] = 0$$

$$e^{i\omega t}\nabla^2\tilde{p} + \frac{\omega^2}{c^2}\tilde{p}\, e^{i\omega t} = 0$$

$$\nabla^2\tilde{p} + \frac{\omega^2}{c^2}\tilde{p} = 0 \quad \text{(Helmholtz equation)}$$

$p(t,\bar{r}) = f_1(\bar{r}) f_2(t)$ Separation of variables

④ In 3D: $\underline{p} = \underline{A}\, e^{i(\omega t - \bar{k}\cdot\bar{r})}$

$$\bar{k} = k_x \hat{I} + k_y \hat{J} + k_z \hat{K}$$

$$\bar{r} = x \hat{I} + y \hat{J} + z \hat{K}$$

$$\bar{k}\cdot\bar{r} = k_x x + k_y y + k_z z$$

What plane waves satisfy the wave equation

$$k_x^2 + k_y^2 + k_z^2 = |\bar{k}|^2 = \frac{\omega^2}{c^2}$$

Lecture 7

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Q1)

$x=0$

Fluid 1 (ρ_1, c_1)	Fluid 2 (ρ_2, c_2)	Give: A, ω, ρ_1, c_1, ρ_2, c_2
$\vec{k}_r = k_1 \hat{i}$ ⟵	$\vec{k}_t = k_2 \hat{i}$ ⟶	
$\underline{P}_i = \underline{A} e^{j(\omega t - k_1 x)}$	$\underline{P}_t = \underline{D} e^{j(\omega t - k_2 x)}$	Find: $\underline{P}_r = ?$, $\underline{P}_t = ?$ ($\underline{B} = ?$) ($\underline{D} = ?$)
⟵ $\vec{k}_r = -k_1 \hat{i}$		
$\underline{P}_r = \underline{B} e^{j(\omega t + k_1 x)}$	Fig 1	

Boundary Conditions

① Components of the wave vector parallel to the interface are the same for the incident, reflected and transmitted waves

$(k_{i,y} = k_{r,y} = k_{t,y} = 0 \;,\; k_{i,z} = k_{r,z} = k_{t,z} = 0)$

② The pressure across the interface is continuous

$\left(\underline{P}_i(x=0) + \underline{P}_r(x=0) = \underline{P}_t(x=0)\right)$

③ The component of the particle velocity in the direction perpendicular to the interface is continuous (valid only at the interfaces between two ~~fluids~~)

$\left(v_{i,x}(x=0) + v_{r,x}(x=0) = v_{t,x}(x=0)\right)$

②: $\underline{P}_i, \underline{P}_r, \underline{P}_t$ at $x=0$ are given $\Rightarrow \underline{A} e^{j\omega t} + \underline{B} e^{j\omega t} = \underline{D} e^{j\omega t} \Rightarrow \boxed{\underline{A} + \underline{B} = \underline{D}}$

③: $\vec{v} = \frac{p}{\rho c}\hat{i}$ from Conservation of momentum $\left(\nabla p + \rho_0 \frac{\partial \vec{v}}{\partial t} = 0\right)$ at $x=0$

$\Rightarrow \frac{\underline{A}}{\rho_1 c_1} e^{j\omega t} - \frac{\underline{B}}{\rho_1 c_1} e^{j\omega t} = \frac{\underline{D}}{\rho_2 c_2} e^{j\omega t} \Rightarrow \boxed{\underline{A} - \underline{B} = \frac{\rho_1 c_1}{\rho_2 c_2}\underline{D}}$

Ans1)

$$\underline{B} = \underline{A}\,\frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1}, \qquad \underline{D} = \underline{A}\,\frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1}$$

Reflection coefficient

$$\Rightarrow \underline{R} \triangleq \left.\frac{\underline{P}_r}{\underline{P}_i}\right|_{@\text{boundary}} = \frac{\underline{B}}{\underline{A}} \;(\underline{P}_r(x=0),\ \underline{P}_i(x=0)) = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1}$$

Transmission coefficient

$$\underline{T} \triangleq \left.\frac{\underline{P}_t}{\underline{P}_i}\right|_{@\text{boundary}} = \frac{\underline{D}}{\underline{A}} \;(\underline{P}_t(x=0),\ \underline{P}_i(x=0)) = \frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1}$$

⟨definition⟩ ⟨only in our example⟩

Lecture 8

For plane wave $\underline{P} = \underline{A} e^{j(\omega t - \vec{k}\cdot\vec{r})}$

$\Rightarrow \vec{\underline{V}} = \hat{I}\underline{V} = \hat{I}\frac{\underline{P}}{\rho c}$ (conservation of momentum (Lec7 page1)) $\Rightarrow \underline{\rho c} = \frac{\underline{P}}{\underline{V}} = \frac{\text{Action}}{\text{Reaction}} = \text{Impedance}$ (constant)

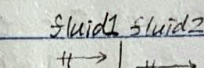
⟨ Characteristic Impedance ⟩

$\underline{Z}_c = \left.\frac{\underline{P}}{\underline{V}}\right|_{\text{for a plane wave}} = \underline{\rho c} = \text{constant parameter of fluid}$

⟨ Specific Impedance ⟩

$\underline{Z}_s = \left.\frac{\underline{P}}{\underline{V}}\right|_{\text{for any arbitary wave}} = \text{Specific to the condition \& location}$

Q2)

Given: $\underline{A}, \omega, \rho_1, c_1, \rho_2, c_2$ in Lec7 Fig1, 

Find: $\underline{Z}_{s1} = ?$ in fluid 1 & $\underline{Z}_{s2} = ?$ in fluid 2

$\Rightarrow$ In fluid 2: $\underline{Z}_{s2} = \frac{\underline{P}_{total,2}}{\underline{V}_{total,2}} = \underline{Z}_{c2} = \rho_2 c_2$ (In fluid 2, there is only a plane wave, so $\underline{Z}_s = \underline{Z}_c$)

In fluid 1: $\underline{Z}_{s1} = \frac{\underline{P}_{total,1}}{\underline{V}_{total,1}} = \frac{\underline{P}_i + \underline{P}_r}{\underline{V}_i - \underline{V}_r}$ (In fluid 1, there are two plane waves, so $\underline{Z}_s \neq \underline{Z}_c$)

$= \frac{\underline{A} e^{j\omega t}(e^{-jk_1 x} + \underline{R} e^{jk_1 x})}{\frac{\underline{A}}{\underline{Z}_{c1}} e^{j\omega t}(e^{-jk_1 x} - \underline{R} e^{jk_1 x})}$

$= \frac{e^{-jk_1 x} + \underline{R} e^{jk_1 x}}{e^{-jk_1 x} - \underline{R} e^{jk_1 x}} \underline{Z}_{c1}$

$= \frac{1 + \underline{R} e^{2jk_1 x}}{1 - \underline{R} e^{2jk_1 x}} \underline{Z}_{c1}$ ← Depends on position and frequency

$\vec{\underline{V}}_i = \underline{V}_i \hat{I} = \frac{\underline{A}}{\underline{Z}_{c1}} e^{j(\omega t - k_1 x)} \hat{I}$

$\vec{\underline{V}}_r = \underline{V}_r \hat{I} = \frac{\underline{R}\underline{A}}{\underline{Z}_{c1}} e^{j(\omega t + k_1 x)}(-\hat{I})$

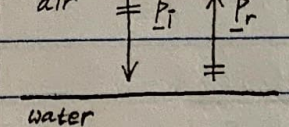
$\underline{V}_{total,1} = \vec{\underline{V}}_i + \vec{\underline{V}}_r = \hat{I}\frac{\underline{A}}{\underline{Z}_{c1}} e^{j\omega t}(e^{-jk_1 x} - \underline{R} e^{jk_1 x})$

$= \hat{I}(\underline{V}_i - \underline{V}_r)$

$\underline{P}_{total,1} = \underline{P}_i + \underline{P}_r = \underline{A} e^{j\omega t}[e^{-jk_1 x} + \underline{R} e^{jk_1 x}]$

cf) velocity: a vector quatity
pressure: a scalar quantity

Q3)



Given: $\rho_{air} = 1.29\ [kg/m^3]$, $c_{air} = 343\ [m/s]$

$\rho_{water} = 1000\ [kg/m^3]$, $c_{water} = 1500\ [m/s]$

Find: $\underline{Z}_{s,air}$

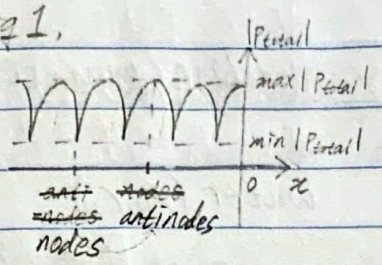
$\underline{R} = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \cong 1$

$\Rightarrow \underline{Z}_{s,air} = \frac{1 + \underline{R} e^{2jk_1 x}}{1 - \underline{R} e^{2jk_1 x}} \cong \frac{1 + e^{2jk_1 x}}{1 - e^{2jk_1 x}} = \frac{e^{-jk_1 x} + e^{jk_1 x}}{e^{-jk_1 x} - e^{jk_1 x}}$

$= \frac{2\cos(k_1 x)}{-2j\sin(k_1 x)} = j \cdot a\tan(k_1 x)$

⟨Standing Wave Ratio⟩

For Fig 1,



Standing Wave Ratio (SWR)

$$SWR = \frac{\max |\underline{P}_{total}|}{\min |\underline{P}_{total}|}$$

Notes
① SWR = 1 for plane wave
② SWR = ∞ for purely standing wave
③ SWR ≥ 1 in general

$$SWR \triangleq \frac{\max |\underline{P}_i + \underline{P}_r|}{\min |\underline{P}_i + \underline{P}_r|} = \frac{\underline{A}(1+R)}{\underline{A}(1-R)} = \frac{1+R}{1-R}$$

$$|\underline{P}_i + \underline{P}_r| = \left| \underline{A} e^{j(\omega t - k_1 x)} + \underline{R}\underline{A} e^{j(\omega t + k_1 x)} \right|$$

$$= |\underline{A}| \left| e^{j\omega t} \right| \left| e^{-jk_1 x} \right| \left| 1 + R e^{j(2k_1 x + \phi)} \right|$$

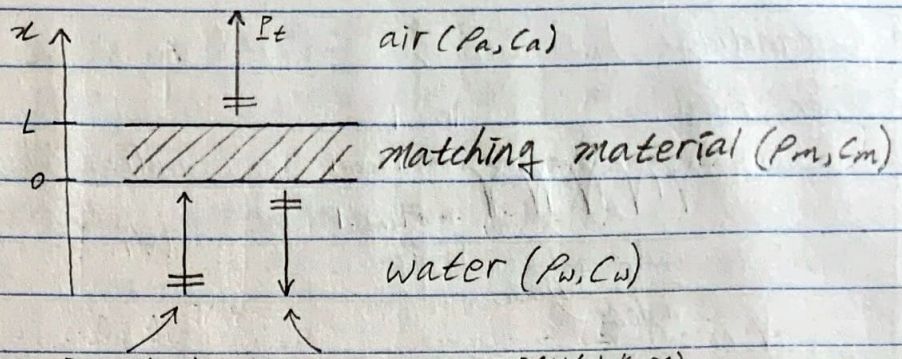
$$= |\underline{A}| \sqrt{[1 + R\cdot\cos(2k_1 x + \phi)]^2 + [R\cdot\sin(2k_1 x + \phi)]^2}$$

$$= |\underline{A}| \sqrt{1 + 2R\cdot\cos(2k_1 + \phi) + R^2}$$

max when $\cos(2k_1 + \phi) = 1 \Rightarrow |\underline{A}|(1+R)$

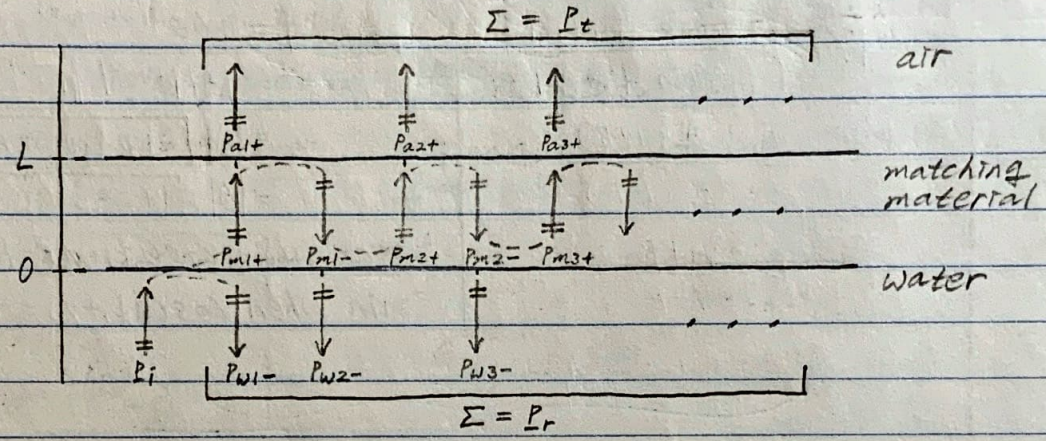
min when $\cos(2k_1 + \phi) = -1 \Rightarrow |\underline{A}|\,|1-R|$

Lecture 9



Q) Given: $L, \underline{A}, \omega, \rho_a, c_a, \rho_w, c_w$

Find: ρ_m, c_m that makes $\underline{P}_r = 0$



* The superposition of plane waves of same frequency and wave vector is a "plane wave"

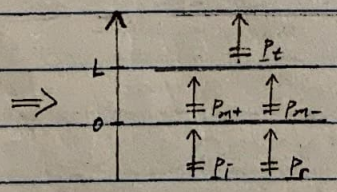
$$\left(\underline{A}_1 e^{j(\omega t - \bar{k}\cdot\bar{r})} + \underline{A}_2 e^{j(\omega t - \bar{k}\cdot\bar{r})} = (\underline{A}_1 + \underline{A}_2) e^{j(\omega t - \bar{k}\cdot\bar{r})} \right)$$

$$\Rightarrow \underline{P}_r = P_{w1-} + P_{w2-} + P_{w3-} + \cdots = \underline{B} e^{j(\omega t + k_w x)}$$

$$\underline{P}_t = P_{a1+} + P_{a2+} + P_{a3+} + \cdots = \underline{D} e^{j(\omega t - k_a x)}$$

$$\underline{P}_{m+} = P_{m1+} + P_{m2+} + P_{m3+} + \cdots = \underline{M} e^{j(\omega t - k_m x)}$$

$$\underline{P}_{m-} = P_{m1-} + P_{m2-} + P_{m3-} + \cdots = \underline{N} e^{j(\omega t + k_m x)}$$



Apply Boundary Conditions to find $\underline{B}, \underline{D}, \underline{M}, \underline{N}$

At $x=0$:

① Pressure conserved

$$\underline{P}_i + \underline{P}_r = \underline{P}_{m+} + \underline{P}_{m-} \Big|_{@x=0} \Rightarrow \boxed{\underline{A} + \underline{B} = \underline{M} + \underline{N}}$$

② Particle velocity component in x is conserved

$$\underline{v}_i + \underline{v}_r = \underline{v}_{m+} + \underline{v}_{m-} \Big|_{@x=0} \Rightarrow \frac{\underline{A}}{Z_w} - \frac{\underline{B}}{Z_w} = \frac{\underline{M}}{Z_m} - \frac{\underline{N}}{Z_m}$$ ← characteristic impedance

$$\Rightarrow \boxed{\frac{Z_m}{Z_w}(\underline{A} - \underline{B}) = \underline{M} - \underline{N}}$$

At $x=L$:

③ Pressure conserved

$$\underline{P}_{m-} + \underline{P}_{m+} = \underline{P}_t \Big|_{@x=L} \Rightarrow \boxed{\underline{M} e^{-jk_m L} + \underline{N} e^{jk_m L} = \underline{D} e^{-jk_a L}}$$

④ Particle velocity component in x is conserved

$$\underline{v}_{m-} + \underline{v}_{m+} = \underline{v}_t \Big|_{@x=L} \Rightarrow \frac{\underline{M}}{Z_m} e^{-jk_m L} - \frac{\underline{N}}{Z_m} e^{jk_m L} = \frac{\underline{D}}{Z_a} e^{-jk_a L}$$

$$\Rightarrow \boxed{\underline{M} e^{-jk_m L} - \underline{N} e^{jk_m L} = \frac{Z_m}{Z_a} \underline{D} e^{-jk_a L}}$$

① + ② $\Rightarrow \underline{M} = \frac{1}{2}(\underline{A} + \underline{B}) + \frac{1}{2}\frac{Z_m}{Z_w}(\underline{A} - \underline{B})$

① − ② $\Rightarrow \underline{N} = \frac{1}{2}(\underline{A} + \underline{B}) - \frac{1}{2}\frac{Z_m}{Z_w}(\underline{A} - \underline{B})$

$$\Rightarrow \frac{\underline{M}}{\underline{N}} = \frac{\underline{A}\left(1 + \frac{Z_m}{Z_w}\right) + \underline{B}\left(1 - \frac{Z_m}{Z_w}\right)}{\underline{A}\left(1 - \frac{Z_m}{Z_w}\right) + \underline{B}\left(1 + \frac{Z_m}{Z_w}\right)}$$

③ + ④ $\Rightarrow \underline{M} e^{-jk_m L} = \frac{1}{2}\left(1 + \frac{Z_m}{Z_a}\right)\underline{D} e^{-jk_a L}$

③ − ④ $\Rightarrow \underline{N} e^{jk_m L} = \frac{1}{2}\left(1 - \frac{Z_m}{Z_a}\right)\underline{D} e^{-jk_a L}$

$$\Rightarrow \frac{\underline{M}}{\underline{N}} = \frac{1 + \frac{Z_m}{Z_a}}{1 - \frac{Z_m}{Z_a}} e^{2jk_m L}$$

Goal: $B = 0 \Rightarrow$

$$\frac{1 + \frac{Z_m}{Z_w}}{1 - \frac{Z_m}{Z_w}} = \frac{1 + \frac{Z_m}{Z_a}}{1 - \frac{Z_m}{Z_a}} e^{2jk_m L}$$

$$\boxed{\frac{Z_w + Z_m}{Z_w - Z_m} = \frac{Z_a + Z_m}{Z_a - Z_m} e^{2jk_m L}} \quad \circledast$$

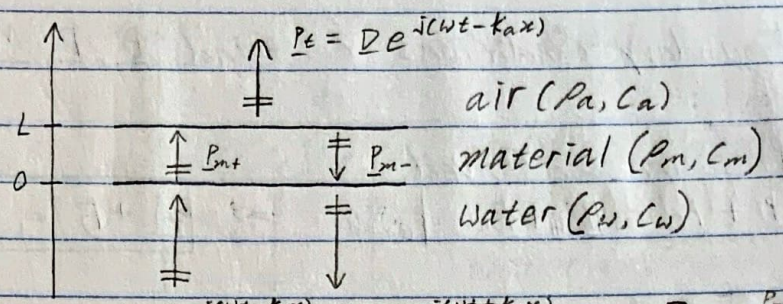
real / real / complex

$\Rightarrow e^{2jk_m L}$ has to be real to satisfy the equation $\Rightarrow 1$ or -1

$\Rightarrow$ Cannot be 1 because $Z_w \neq Z_a \Rightarrow e^{2jk_m L} = -1 \Rightarrow k_m = \frac{(2n+1)\pi}{2L}$

$\Rightarrow C_m = \frac{\omega}{k_m} = \frac{2L\omega}{(2n+1)\pi}$ & $\rho_m = \frac{Z_m}{C_m} = \frac{\sqrt{Z_a Z_w}}{C_m}$ ← from $\circledast$ & $e^{2jk_m L} = -1$

Lecture 10



$$\Rightarrow \underline{R} = \left.\frac{P_r}{P_i}\right|_{@x=0} = \frac{B}{A}$$

Derived $\rightsquigarrow$

$$\frac{Z_w + Z_m + \underline{R}(Z_w - Z_m)}{Z_w - Z_m + \underline{R}(Z_w + Z_m)} = \frac{Z_a + Z_m}{Z_a - Z_m} e^{2jk_m L}$$

$\underline{R} = 0 \Rightarrow$

$$e^{2jk_m L} = -1 \Rightarrow Z_m = \sqrt{Z_a Z_w}$$

$$\hookrightarrow 2\frac{\omega}{c_m} L = (2n+1)\pi \Rightarrow c_m = \frac{2 \cdot 2\pi f \cdot L}{(2n+1)\pi} = \frac{4Lf}{2n+1}$$

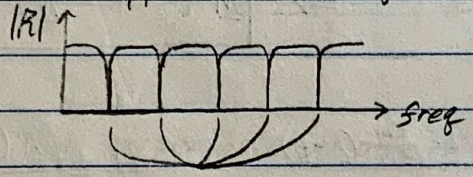
For $n=0$: $c_m = 4Lf \Rightarrow L = \frac{c_m}{4f} = \frac{\lambda_m}{4}$

$\Rightarrow$ Quater wavelength matching layers

(Condition: $Z_m = \sqrt{Z_a Z_w}$, $L = \frac{\lambda_m}{4}$

Result: Reflection $= \underline{0}$

- What happens if frequency changes?



Sound is transmitted

* Reflection coefficient at $x=0$

BCs at $x=0$:

① $\underline{A} e^{j\omega t} + \underline{B} e^{j\omega t} = P_{m+} + P_{m-} = P_m$ (total pressure)

② $\frac{\underline{A} e^{j\omega t}}{Z_w} - \frac{\underline{B} e^{j\omega t}}{Z_w} = \underline{v}_{m+} + \underline{v}_{m-} = \underline{v}_m$ (total particle velocity)

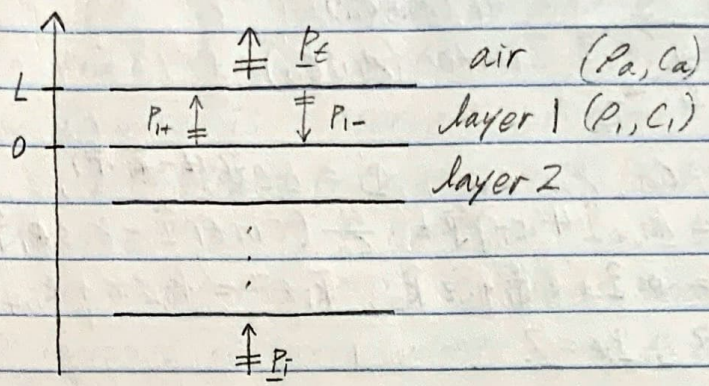
$$\Rightarrow Z_w \frac{\underline{A} e^{j\omega t} + \underline{B} e^{j\omega t}}{\underline{A} e^{j\omega t} - \underline{B} e^{j\omega t}} = \underline{Z}_s(x=0)$$

$$\Rightarrow Z_w \frac{1+\underline{R}}{1-\underline{R}} = \underline{Z}_s(x=0) \Rightarrow \underline{R} = \frac{\underline{Z}_s(x=0) - Z_w}{\underline{Z}_s(x=0) + Z_w}$$

Q)

Given: $\rho_a, c_a, \rho_1, c_1, L$

Find: $\underline{Z}_s(x=0) = ?$



$$\Rightarrow \underline{P}_{1+} = \underline{E}\, e^{j(\omega t - k_1 x)}$$
$$\underline{P}_{1-} = \underline{G}\, e^{j(\omega t + k_1 x)}$$
$$\underline{P}_t = \underline{D}\, e^{j(\omega t - k_a x)}$$

$$\Rightarrow \underline{Z}_s(x=0) = \frac{\underline{P}_{1+}(x=0) + \underline{P}_{1-}(x=0)}{\frac{\underline{P}_{1+}(x=0)}{Z_1} - \frac{\underline{P}_{1-}(x=0)}{Z_1}} = \frac{\underline{E}\, e^{j\omega t} + \underline{G}\, e^{j\omega t}}{\underline{E}\, e^{j\omega t} - \underline{G} e^{j\omega t}} Z_1$$

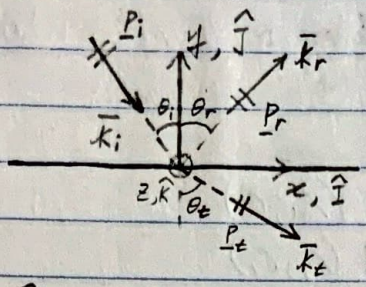
$$= Z_1 \frac{1 + \underline{G}/\underline{E}}{1 - \underline{G}/\underline{E}}$$

$$\frac{\underline{P}_{1-}(x=L)}{\underline{P}_{1+}(x=L)} = \frac{Z_a - Z_1}{Z_a + Z_1} = \frac{\underline{G}\, e^{j(\omega t + k_1 L)}}{\underline{E}\, e^{j(\omega t - k_1 L)}} = \frac{\underline{G}}{\underline{E}} e^{2jk_1 L}$$

$$\Rightarrow \frac{\underline{G}}{\underline{E}} = \frac{Z_a - Z_1}{Z_a + Z_1} e^{-2jk_1 L}$$

$$\therefore Z_s(x=0) = Z_1 \frac{1 + \frac{Z_a - Z_1}{Z_a + Z_1} e^{-2jk_1 L}}{1 - \frac{Z_a - Z_1}{Z_a + Z_1} e^{-2jk_1 L}}$$

Lecture 11



air (ρ_a, c_a)

Water (ρ_w, c_w)

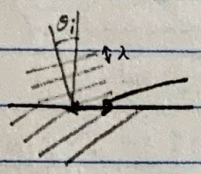
Given: ρ_a, c_a, ρ_w, c_w, $\underline{P}_i = \underline{A} e^{j(\omega t - \bar{k}_i \cdot \bar{r})}$,

$\bar{k}_i = k_{i,x}\hat{I} + k_{i,y}\hat{J} = \frac{\omega}{c_a}(\sin\theta_i \hat{I} - \cos\theta_i \hat{J})$,

$\bar{r} = x\hat{I} + y\hat{J} + z\hat{K}$, $\bar{k}_i \cdot \bar{r} = k_{i,x}x + k_{i,y}y$

Find: $\underline{P}_r = ?$, $\underline{P}_t = ?$

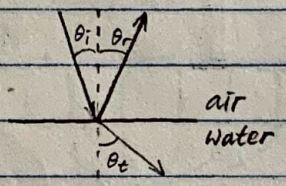
(1)



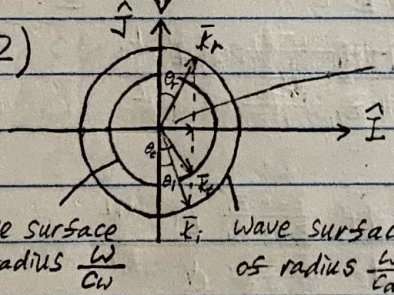
$$d = \frac{\lambda}{\sin\theta_i} = \frac{2\pi/k}{\sin\theta_i} = \frac{2\pi}{k_{i,x}} = \frac{2\pi}{k_{r,x}} = \frac{2\pi}{k_{t,x}}$$

$$\Rightarrow k_{i,x} = k_{r,x} = k_{t,x}$$

The components of the incident, reflected and transmitted wave vectors in the direction parallel to the interface are conserved.



(2)

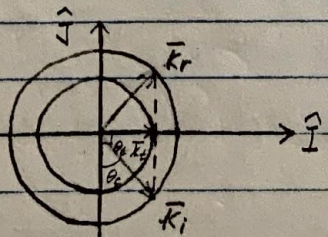


$$k_{i,x} = k_{r,x} = k_{t,x} \Rightarrow \frac{\omega}{c_a}\sin\theta_i = \frac{\omega}{c_a}\sin\theta_r = \frac{\omega}{c_w}\sin\theta_t$$

$\Rightarrow$ **Snell's law**

$$\begin{cases} \theta_i = \theta_r \\ \frac{\sin\theta_i}{c_a} = \frac{\sin\theta_t}{c_w} \end{cases}$$

(3)



From Snell's law

$$\therefore \frac{\sin\theta_c}{c_a} = \frac{\sin 90°}{c_w} = 1 \Rightarrow \sin\theta_c = \frac{c_a}{c_w}$$

$\Rightarrow$ **Critical angle**

$$\theta_c = \text{asin}\frac{c_a}{c_w}$$

For $\theta_i > \theta_c$, no transmitted wave in the water (perfect reflection)

$\underline{P}_r = \underline{B}\, e^{j(\omega t - \bar{k}_r \cdot \bar{r})}$ ← Need to find $\underline{B}$

$\bar{k}_r = k_{r,x}\hat{I} + k_{r,y}\hat{J} = \frac{\omega}{c_a}(\sin\theta_i \hat{I} + \cos\theta_i \hat{J})$

$\underline{P}_t = \underline{D}\, e^{j(\omega t - \bar{k}_t \cdot \bar{r})}$ ← Need to find $\underline{D}$

$\bar{k}_t = k_{t,x}\hat{I} + k_{t,y}\hat{J} = \frac{\omega}{c_w}(\sin\theta_t \hat{I} + \cos\theta_t \hat{J})$

① Pressure is conserved at $y=0$

$\underline{P}_i(x, y=0) + \underline{P}_r(x, y=0) = \underline{P}_t(x, y=0)$

$\underline{A}\, e^{j\omega t} e^{-jk_{i,x}x} + \underline{B} e^{j\omega t} e^{-jk_{r,x}x} = \underline{D}\, e^{j\omega t} e^{-jk_{t,x}x}$

$\Rightarrow \boxed{\underline{A} + \underline{B} = \underline{D}}$ (Same as normal incidence case)

Lecture 12

② Component of particle velocity normal to the interface is conserved

$\underline{V}_{i,y}(x, y=0) + \underline{V}_{r,y}(x, y=0) = \underline{V}_{t,y}(x, y=0)$

$\underline{\bar{V}}_i\Big|_{y=0} = \frac{\underline{P}_i}{Z_a}\hat{k}_i = \frac{\underline{A}}{Z_a} e^{j(\omega t - k_{i,x}x)}(\sin\theta_i \hat{I} - \cos\theta_i \hat{J})$

$\hat{k}_i$: unit vector with direction of $\bar{k}_i$

normal to the interface $\Rightarrow$

$\frac{\underline{A}}{Z_a} e^{j(\omega t - k_{i,x}x)}(-\cos\theta_i) + \frac{\underline{B}}{Z_a} e^{j(\omega t - k_{r,x}x)}\cos\theta_i$

$= \frac{\underline{D}}{Z_w} e^{j(\omega t - k_{t,x}x)}(-\cos\theta_t)$

$\Rightarrow \frac{\underline{A}}{Z_a}\cos\theta_i - \frac{\underline{B}}{Z_a}\cos\theta_i = \frac{\underline{D}}{Z_w}\cos\theta_t \Rightarrow \boxed{\frac{\underline{A}}{Z_a'} - \frac{\underline{B}}{Z_a'} = \frac{\underline{D}}{Z_w'}}$ $Z_a' = Z_a/\cos\theta_i$, $Z_w' = Z_w/\cos\theta_t$

(Same form as for normal incidence)

$\therefore \boxed{\underline{B} = \underline{A}\frac{Z_w' - Z_a'}{Z_w' + Z_a'}\ ;\ \underline{D} = \underline{A}\frac{2Z_w'}{Z_w' + Z_a'}}$

(cf) Reflection coefficient: $\underline{R} \triangleq \frac{\underline{P}_r}{\underline{P}_i}\Big|_{@y=0}$

$\underline{R} = \frac{\underline{B}}{\underline{A}} = \frac{Z_w' - Z_a'}{Z_w' + Z_a'}$

Transmission coefficient: $\underline{T} \triangleq \frac{\underline{P}_t}{\underline{P}_i}\Big|_{@y=0}$

$\underline{T} = \frac{\underline{D}}{\underline{A}} = \frac{2Z_w'}{Z_w' + Z_a'}$

Lecture 12

42:53 ⟨Evanescent waves⟩

$\bar{k}_t = k_{i,x}\hat{I} + k_{t,y}\hat{J} \Rightarrow |\bar{k}_t|^2 = k_{i,x}^2 + k_{t,y}^2 = \left(\frac{\omega}{c_w}\right)^2$

$\Rightarrow k_{t,y} = \pm\sqrt{\left(\frac{\omega}{c_w}\right)^2 - k_{i,x}^2}$ ↓ If $\theta_c < \theta_i$, then $\left(\frac{\omega}{c_w}\right)^2 < k_{i,x}^2$

$= \pm j\sqrt{k_{i,x}^2 - \left(\frac{\omega}{c_w}\right)^2}$

$= \alpha$ (α real & positive value)

⟨α⟩

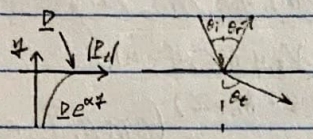
$\Rightarrow \bar{k}_t = k_{i,x}\hat{I} \pm j\alpha\hat{J}$

$\underline{P}_t = \underline{D}\, e^{j(\omega t - k_{i,x}x \pm j\alpha y)} = \underbrace{\underline{D}\, e^{\mp\alpha y}}_{|P_t|}\ \underbrace{e^{j(\omega t - k_{i,x}x)}}_{\text{Propagation in } +x \text{ direction}}$

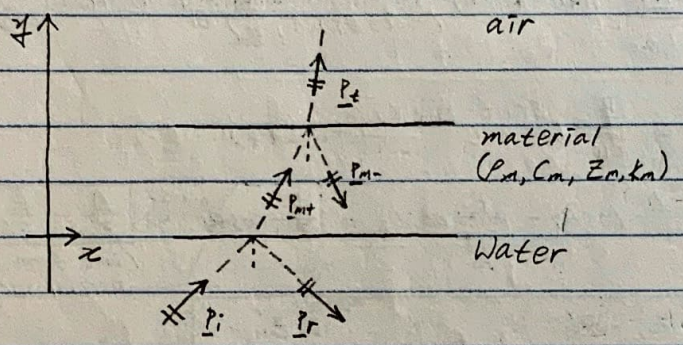
↳ exponential decay in the $-y$ direction

$\Rightarrow \underline{P}_t = \underline{D}\, e^{\alpha y} e^{j(\omega t - k_{i,x}x)}$ ← value

($\bar{k}_t = k_{i,x}\hat{I} - j\alpha\hat{J}$)



⟨Multiple media⟩



$\underline{P}_i = \underline{A}\, e^{j(\omega t - k_{i,x}x)}\, e^{-jk_{i,y}y}$

* Quarter wavelength matching layer

Normal incidence : $Z_m = \sqrt{Z_a Z_w}$, $k_m = \frac{\pi}{2L}$

Oblique incidence : $Z'_m = \sqrt{Z'_a Z'_w}$, $k_{m,y} = \frac{\pi}{2L}$

Lecture 13

$p = p_0 + p$ ← acoustic pressure

$\rho = \rho_0 + \rho_0 s$ ← condensation

$\bar{v}$

↓

Eq. of state: $p = Bs$ ← bulk modulus

"Linearized equations"

Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 \rightarrow \frac{\partial s}{\partial t} + \nabla \cdot \bar{v} = 0$

Conservation of momentum: $\nabla p + \frac{\partial (\rho \bar{v})}{\partial t} = 0 \rightarrow \nabla p + \rho_0 \frac{\partial \bar{v}}{\partial t} = 0$

↓

Wave equation: $\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$ $\left(c^2 = \frac{B}{\rho_0}\right)$

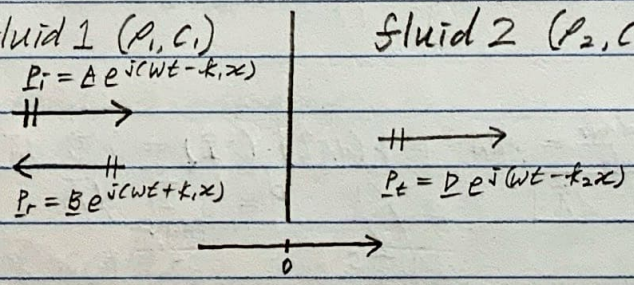
General solution: $\underline{p} = \underline{A} e^{j(\omega t - \bar{k} \cdot \bar{r})}$ (plane wave)

$|\bar{k}| = \frac{\omega}{c}$ (wave number)

Particle velocity of plane wave: $\bar{v} = \underline{v} \hat{k} = \frac{\underline{p}}{\rho_0 c} \hat{k}$ ← unit vector in the direction of $\bar{k}$

↓

Interface between 2 semi-infinite fluids



① Component of $\bar{k}$ parallel to the interface is the same for $\underline{p_i}, \underline{p_r}, \underline{p_t}$

② Pressure is conserved across the interface

③ Component of $\bar{v}$ perpendicular on the interface is conserved

↓

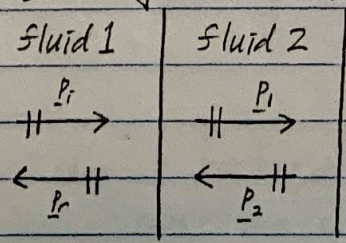
Reflection coefficient: $\underline{R} = \frac{\underline{p_r}}{\underline{p_i}}\Big|_{@\text{ interface}}$

Transmission coefficient: $\underline{T} = \frac{\underline{p_t}}{\underline{p_i}}\Big|_{@\text{ interface}}$

Characteristic impedance: $\underline{Z_c} = \frac{\underline{p}}{\underline{v}}\Big|_{\text{for a plane wave}} = \rho_0 c = \frac{\underline{p}}{\bar{v} \cdot \hat{k}}$

Specific impedance: $\underline{Z_s} = \frac{\underline{p}}{\underline{v}}\Big|_{\text{for the specific wave}}$

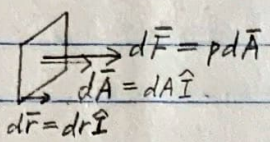
Standing wave ratio: $\frac{\max |p_{total}|}{\min |p_{total}|} = SWR$



$$\underline{R} = \frac{\underline{Z_{s2}}(x=0) - \underline{Z_1}}{\underline{Z_{s2}}(x=0) + \underline{Z_1}}$$

Lecture 14

Very small surface described by $d\bar{A}$ (direction: perpendicular to the surface, magnitude: area of the surface)



Track $d\bar{A}$ over dt

$\Rightarrow$ Work done by $d\bar{F}$ through $d\bar{r}$

: $dW = d\bar{F} \cdot d\bar{r} = p\,dA\,dr$

$p = Re\{\tilde{p}\}$

⟨Intensity⟩

① Instantaneous intensity

$\bar{I}_i$ → magnitude : $|I_i| = \frac{dW}{dA\,dt} = \frac{p\,dA\,dr}{dA\,dt} = p\frac{dr}{dt} = pv$ (v: velocity of the surface)

→ direction : $\hat{k}$

$\Rightarrow \boxed{\bar{I}_i = p\bar{v}}$

Notes: ① unit : $\left[\frac{Pa \cdot m}{s}\right] = \left[\frac{W}{m^2}\right]$

② Intensity is a vector

② Acoustic (average) intensity (What instruments measure)

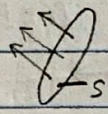
$\bar{I} = \frac{1}{T}\int_0^T \bar{I}_i\,dt$

↳ usually long time interval

⟨Power⟩

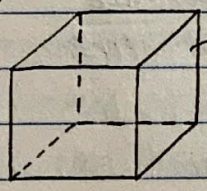
$P \triangleq \frac{dW}{dt} = \frac{p\,dA\,dr}{dt} = p\,dA\,v = p(dA\hat{I}) \cdot (v\hat{I}) = (p\bar{v}) \cdot d\bar{A}$

$\Rightarrow \boxed{P = \bar{I} \cdot d\bar{A}}$



$P = \int_{surfaces S} \bar{I}_i \cdot d\bar{A}$

⟨Energy⟩



volumn : dV

mass : $dm = \rho dV$ $\left(\rho = \frac{dm}{dV}\right)$

de_T = kinetic energy + potential energy

$= \frac{1}{2}\,dm\,|\bar{v}|^2 + \frac{1}{2}\,dm\,\frac{p^2}{\rho^2 c^2}$

$= \frac{1}{2}\,dm\left(|\bar{v}|^2 + \frac{p^2}{\rho^2 c^2}\right)$

⟨Energy density⟩

$E_i = \frac{de_T}{dV} = \frac{1}{2}\rho_0\left(|\bar{v}|^2 + \frac{p^2}{\rho_0^2 c^2}\right)$

Example

$\underline{p} = \underline{A} e^{j(\omega t - kx)}$

$= A e^{j(\omega t - kx + \phi)}$

Given: $\rho_0, c, \underline{A}, \omega$

Find: $\bar{I}_i, \bar{I}, E_i$

$p = Re\{\underline{p}\} = A\cos(\omega t - kx + \phi)$

$\underline{v} = \frac{\underline{p}}{\rho_0 c}\hat{I} \Rightarrow \bar{v} = Re\{\underline{v}\} = Re\left\{\frac{A e^{j(\omega t - kx)}}{\rho_0 c}\hat{I}\right\}$

$= \frac{A\cos(\omega t - kx + \phi)}{\rho_0 c}\hat{I}$

$\Rightarrow \bar{I}_i = p\bar{v} = \frac{A^2}{\rho_0 c}\cos^2(\omega t - kx + \phi)\hat{I}$

$\bar{I} = \frac{1}{T}\int_0^T \bar{I}_i\, dt = \frac{1}{T}\int_0^T \frac{A^2}{\rho_0 c}\cos^2(\omega t - kx + \phi)\, dt\, \hat{I}$

$= \frac{A^2}{\rho_0 c}\frac{1}{T}\int_0^T \cos^2(\omega t - kx + \phi)\, dt\, \hat{I}$

$= \frac{1}{2}\frac{A^2}{\rho_0 c}\hat{I}$ $\left(\frac{1}{T}\int_0^T \cos^2(\omega t - kx + \phi)\, dt = \frac{1}{2}\right)$ $\left(\begin{matrix}\cos(x+y) = \cos x\cos y - \sin x \sin y \\ \sin(x+y) = \sin x \cos y + \cos x \sin y\end{matrix}\right)$

$E_i = \frac{1}{2}\rho_0\left(|\bar{v}|^2 + \frac{p^2}{\rho_0^2 c^2}\right)$

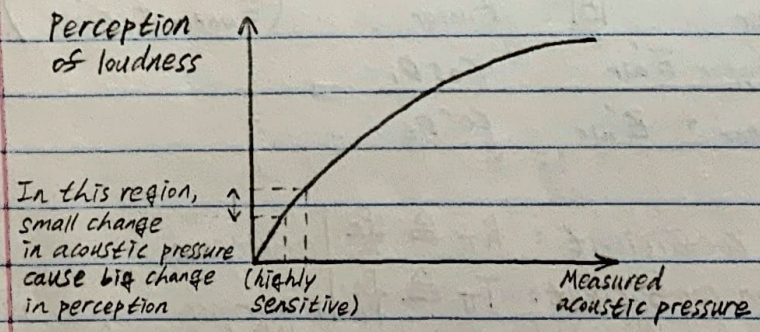
$= \frac{1}{2}\rho_0\left(\frac{A^2\cos^2(\omega t - kx + \phi)}{\rho_0^2 c^2} + \frac{A^2\cos^2(\omega t - kx + \phi)}{\rho_0^2 c^2}\right)$

$= \frac{A^2}{\rho_0 c^2}\cos^2(\omega t - kx + \phi)$

⟨Conservation of energy⟩

$\frac{\partial E_i}{\partial t} + \nabla \cdot \bar{I}_i = 0$ (cf) Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{v}) = 0$

⟨Decibel scale⟩

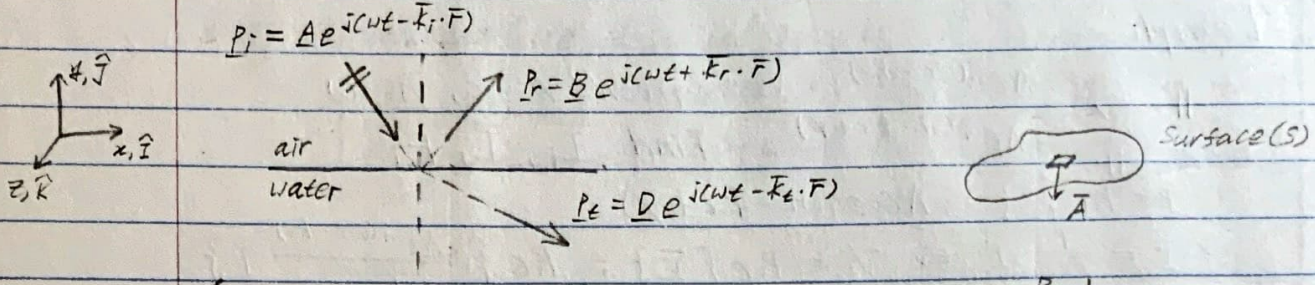


① Intensity level: $IL = 10\log_{10}\left(\frac{|\bar{I}|}{I_{ref}}\right)$ $\left(I_{ref} = 10^{-12}\frac{W}{m^2}\text{ in air};\ [dB\ re\ 10^{-12}\frac{W}{m^2}]\right)$ ← relative

② Sound pressure level: $SPL = 20\log_{10}\left(\frac{P_{rms}}{P_{ref}}\right)$ ($P_{ref} = 20\,\mu Pa$ in air, $P_{ref} = 1\,\mu Pa$ in water, $[dB\ re\ 20\mu Pa]$)

$P_{rms} = \sqrt{\frac{1}{T}\int_0^T p^2\, dt}$ (P_{rms}: root-mean-square; p^2: $Re\{I\}$)

Lecture 15



Pressure reflection coefficient: $\underline{R} \triangleq \left.\frac{\underline{p}_r}{\underline{p}_i}\right|_{@\text{interface}}$

Pressure transmission coefficient: $\underline{T} \triangleq \left.\frac{\underline{p}_t}{\underline{p}_i}\right|_{@\text{interface}}$

Intensity reflection coefficient: $R_I \triangleq \left.\frac{|\bar{I}_r|}{|\bar{I}_i|}\right|_{@\text{interface}}$

Intensity transmission coefficient: $T_I \triangleq \left.\frac{|\bar{I}_t|}{|\bar{I}_i|}\right|_{@\text{interface}}$

Notes: ① R_I and T_I are positive real values

② They are defined in terms of average intensities

$$\underline{R} = \frac{Z'_{water} - Z'_{air}}{Z'_{water} + Z'_{air}}, \quad \underline{T} = \frac{2Z'_{water}}{Z'_{water} + Z'_{air}} \qquad \left(Z'_{water} = \frac{Z_{water}}{\cos\theta_t},\ Z'_{air} = \frac{Z_{air}}{\cos\theta_i}\right)$$

For plane waves, $\bar{I} = p\bar{v} = \frac{|A|^2}{2Z_c}\hat{k}$

$$R_I = \frac{|\underline{B}|^2/Z_{air}}{|\underline{A}|^2/Z_{air}} = \frac{|\underline{B}|^2}{|\underline{A}|^2} = |\underline{R}|^2 = \left|\frac{Z'_{water} - Z'_{air}}{Z'_{water} + Z'_{air}}\right|^2$$

$$T_I = \frac{|\underline{D}|^2/Z_{water}}{|\underline{A}|^2/Z_{air}} = \frac{|\underline{D}|^2}{|\underline{A}|^2}\frac{Z_{air}}{Z_{water}} = \left(\frac{2Z'_{water}}{Z'_{water} + Z'_{air}}\right)^2 \frac{Z'_{air}\cos\theta_i}{Z'_{water}\cos\theta_t}$$

$$= \frac{2Z'_{water}Z'_{air}}{Z'^2_{water} + Z'^2_{air}}\frac{\cos\theta_i}{\cos\theta_t}$$

Power reflection coefficient: $R_\pi \triangleq \left.\frac{P_r}{P_i}\right|_{@\text{interface}}$

Power transmission coefficient: $T_\pi \triangleq \left.\frac{P_t}{P_i}\right|_{@\text{interface}}$

$$\left(P = \int_S \bar{I}\cdot d\bar{A} = \int_S \left(\frac{|A|^2}{2Z_c}\hat{k}\right)\cdot\left[dA(-\hat{j})\right] = \frac{|A|^2}{2Z_c}(-\hat{k}\cdot\hat{j})\int_S dA = \frac{|A|^2 S}{2Z_c}\cos\theta\right)$$

(S = area of the surface)

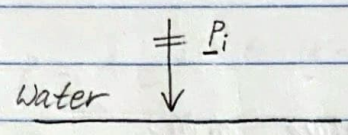
$$R_\pi = \frac{\frac{|B|^2 S}{2Z_{air}}\cos\theta_i}{\frac{|A|^2 S}{2Z_{air}}\cos\theta_i} = \frac{|\underline{B}|^2}{|\underline{A}|^2} = |\underline{R}|^2 = \left(\frac{Z'_{water} - Z'_{air}}{Z'_{water} + Z'_{air}}\right)^2 \approx 1$$

$$T_\pi = \frac{\frac{|D|^2 S}{2Z_{water}}\cos\theta_t}{\frac{|A|^2 S}{2Z_{air}}\cos\theta_i} = \frac{\frac{|\underline{D}|^2}{Z'_{water}}}{\frac{|\underline{A}|^2}{Z'_{air}}} = |\underline{T}|^2\frac{Z'_{air}}{Z'_{water}} = \left(\frac{2Z'_{water}}{Z'_{water} + Z'_{air}}\right)\frac{Z'_{air}}{Z'_{water}}$$

$$= \frac{4Z'_{water}Z'_{air}}{(Z'_{water} + Z'_{air})^2} \approx 0$$

Notes: ① R_π, T_π are always real, positive

② $R_\pi + T_\pi = 1$



Given: $\rho_w = 1000\ kg/m^3$, $C_w = 1500\ m/s$,
$\rho_s = 8000\ kg/m^3$, $C_s = 3200\ m/s$

Find: $\underline{R}$, $\underline{T}$, R_I, T_I, R_π, T_π

$$Z_{water} = 1.5 \times 10^6 \ \frac{s \cdot Pa}{m}, \quad Z_{steel} = 25.6 \times 10^6 \ \frac{s \cdot Pa}{m}$$

$$\underline{R} = \frac{Z_{steel} - Z_{water}}{Z_{steel} + Z_{water}} \cong 0.89$$

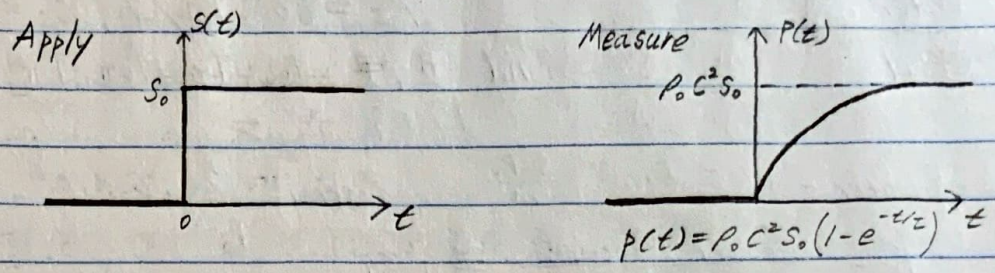
$$\underline{T} = \frac{2\, Z_{steel}}{Z_{steel} + Z_{water}} \cong 1.89$$

$$R_I = |\underline{R}|^2 \cong 0.79, \quad T_I = |\underline{T}|^2 \frac{Z_{water}}{Z_{steel}} \cong 0.21$$

$$R_\pi = |\underline{R}|^2 \cong 0.79, \quad T_\pi \cong 0.21$$

Lecture 16

Eq of state: $P = \rho_0 c^2 s^{\frac{\rho - \rho_0}{\rho_0}}$

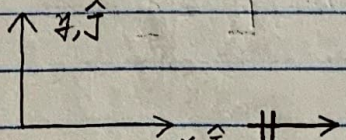


τ : relaxation time

- for gases : τ on order of 10^{-10} seconds
- for liquids : τ on order of 10^{-12} seconds

$$\Rightarrow \begin{cases} \text{Eq of state: } \left(1 + \tau \frac{\partial}{\partial t}\right) P(t) = \rho_0 c^2 s(t) \\ \text{Conservation of mass: } \frac{\partial s}{\partial t} + \nabla \cdot \bar{v} = 0 \\ \text{Conservation of momentum: } \nabla p + \rho \frac{\partial \bar{v}}{\partial t} = 0 \end{cases}$$

$$\Rightarrow \text{Wave equation: } \left(1 + \tau \frac{\partial}{\partial t}\right) \nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}$$



$\bar{k} = k\hat{I}$

$\bar{r} = x\hat{I} + y\hat{J} + z\hat{K}$

$$\underline{p} = \underline{A} e^{j(\omega t - \bar{k} \cdot \bar{r})} = \underline{A} e^{j(\omega t - kx)}$$

$$\Rightarrow \begin{cases} \nabla^2 p = \frac{\partial^2 p}{\partial x^2} = (-jk)^2 \underline{p} = -k^2 \underline{p} \\ \frac{\partial^2 p}{\partial t^2} = (j\omega)^2 \underline{p} = -\omega^2 \underline{p} \end{cases}$$

$$\Rightarrow \left(1 + \tau \frac{\partial}{\partial t}\right)(-k^2 \underline{p}) = \frac{-\omega^2}{c^2} \underline{p}$$

$$-k^2 \underline{p} - k^2 \tau j\omega \underline{p} + \frac{\omega^2}{c^2} \underline{p} = 0$$

$$k^2 = \frac{\omega^2/c^2}{1 + j\omega\tau} \Rightarrow \boxed{\underline{k} = \frac{\omega}{c} \frac{1}{\sqrt{1 + j\omega\tau}}}$$

α : absorption coefficient $\qquad = k' - j\alpha$ (α: real, positive value)

$$\underline{P} = \underline{A}\, e^{j[\omega t - (k' - j\alpha)x]} = \underbrace{\underline{A} e^{-\alpha x}}\; e^{-j(\omega t - k'x)}$$

↳ magnitude of $|\underline{P}|$ decreases as the plane wave propagates in $+x$ direction

Notes: ① The speed of sound in a lossy fluid.

c_p (phase speed = speed of sound)

$$c_p = \frac{\omega}{k'} \neq c \text{ (in general)}$$

② For low loss fluids ($\omega\tau \ll 1$) (cf) $\sqrt{1+x} \cong 1 + \frac{1}{2}x$

$$\sqrt{1 + j\omega\tau} \cong 1 + \frac{j\omega\tau}{2}$$

$$\Rightarrow \underline{k} = \frac{\omega}{c}\,\frac{1}{1 + \frac{j\omega\tau}{2}} = \frac{\omega}{c}\,\frac{1 - j\frac{\omega\tau}{2}}{1^2 + \underbrace{\left(\frac{\omega\tau}{2}\right)^2}_{\approx 0}}$$

$$= \underbrace{\frac{\omega}{c}}_{k'} - j\underbrace{\frac{\omega^2\tau}{2c}}_{\alpha}$$

$$\Rightarrow c_p = c\,;\ \alpha = \frac{\omega^2\tau}{2c} \quad \text{Only for low loss!!}$$

③ Define complex speed of sound: $\underline{c} = \frac{\omega}{\underline{k}}$

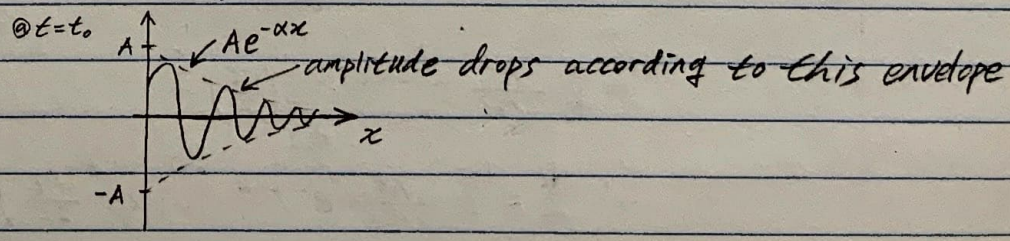
$$\Rightarrow \underline{P} = \underline{A}\, e^{j(\omega t - \frac{\omega}{\underline{c}}x)}$$

All the previous derivations for the characteristic impedance, reflection & transmission coefficients remain the same. We only need to replace c with $\underline{c}$.

* $\underline{c} = c\sqrt{1 + j\omega\tau}$
* $\underline{Z_c} = \rho_0 \underline{c}$
* $\underline{\vec{v}} = \frac{\underline{P}}{\underline{Z_c}}\hat{k}$

④ Plane wave

$$p = \mathrm{Re}\{\underline{P}\} = A e^{-\alpha x}\cos(\omega t - k'x + \phi)$$



Lecture 17

If there is loss,

$\underline{c} = c\sqrt{1+j\omega\tau}$

$\underline{k} = \hat{k}\underline{k} = \hat{k}\frac{\omega}{\underline{c}} = \frac{\omega}{c}\frac{1}{\sqrt{1+j\omega\tau}}\hat{k}$

$\underline{Z_c} = \rho_0\underline{c} = \rho_0 c\sqrt{1+j\omega\tau}$

$\underline{p} = \underline{A}e^{j(\omega t - \underline{k}\cdot\bar{r})}$

$\bar{v} = \frac{\underline{p}}{\underline{Z_c}}\hat{k}$

$\underline{k} = \frac{\omega}{c}\frac{1}{\sqrt{1+j\omega\tau}}$ (from wave equation)

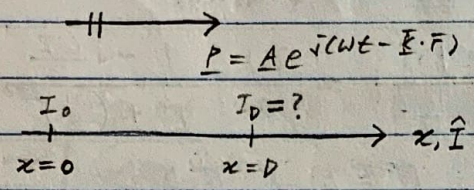
$\Rightarrow \underline{c} = c\sqrt{1+j\omega\tau}$

$\underline{Z_c} = \rho_0\underline{c} = \rho_0 c\sqrt{1+j\omega\tau}$

$\underline{p} = \underline{A}e^{j(\omega t - (\underline{k}\hat{k})\cdot\bar{r})}$

$\bar{v} = \frac{\underline{p}}{\underline{Z_c}}\hat{k}$

⟨Example 1⟩



Given: air ($\rho_0 = 1.29\ kg/m^3$, $c = 343\ m/s$)

$f = 10kHz$, $D = 1\ km$, $I_0 = |\bar{I}(x=0)| = 1\ W/m^2$

Find: $I_D = ?$ when a) $\tau = 0s$

b) $\tau = 200ps$

Step 1: Instantaneous intensity: $\bar{I}_i = p\bar{v}$

Step 2: Acoustic intensity: $\bar{I} = \frac{1}{T}\int_0^T \bar{I}_i\, dt$

$\underline{p} = \underline{A}e^{-\alpha x}e^{j(\omega t - k'x)} = Ae^{-\alpha x}e^{j(\omega t - k'x + \phi)}$

$p = Re\{\underline{p}\} = Ae^{-\alpha x}\cos(\omega t - k'x + \phi)$

$v = Re\{\underline{v}\} = Re\left\{\frac{\underline{p}}{\underline{Z_c}}\right\} = Re\left\{\frac{Ae^{-\alpha x}e^{j(\omega t - k'x+\phi)}}{\rho_0 c\sqrt{1+j\omega\tau}}\right\}$

$|a+jb| = \sqrt{a^2+b^2}$

$\angle(a+jb) = \tan^{-1}\frac{b}{a}$

$\sqrt{1+j\frac{\omega\tau}{2}} = \sqrt{1+\left(\frac{\omega\tau}{2}\right)^2}\,e^{j\phi} = e^{j\frac{\omega\tau}{2}}$

$\cong 1+j\frac{\omega\tau}{2}$ (if $\omega\tau \ll 1$)

$\rho \cong \tan\rho = \frac{\omega\tau}{2} \ll 1$

$\Rightarrow \alpha = \frac{\omega^2\tau}{2c}$

$= \frac{Ae^{-\alpha x}}{\rho_0 c}\cos\left(\omega t - k'x + \phi - \frac{\omega\tau}{2}\right)$

$\Rightarrow \bar{I}_i = \hat{I}\frac{A^2}{\rho_0 c}e^{-2\alpha x}\cos(\omega t - k'x + \phi)\cos\left(\omega t - k'x + \phi - \frac{\omega\tau}{2}\right)$

$\cos a\cos b = \frac{\cos(a+b)+\cos(a-b)}{2}$

$= \hat{I}\frac{A^2}{\rho_0 c}e^{-2\alpha x}\frac{1}{2}\left[\cos\left(2\omega t - 2k'x + 2\phi - \frac{\omega\tau}{2}\right) + \cos\left(\frac{\omega\tau}{2}\right)\right]$

$\Rightarrow \bar{I} = \hat{I}\frac{A^2}{2\rho_0 c}e^{-2\alpha x}\left[\frac{1}{T}\int_0^T \cos(2\omega t - \cdots)\,dt + \frac{1}{T}\int_0^T \cos\left(\frac{\omega\tau}{2}\right)dt\right]$

$\frac{1}{T}\int_0^T \cos(2\omega t - \cdots)\,dt \to 0$

$\frac{1}{T}\int_0^T \cos\left(\frac{\omega\tau}{2}\right)dt = \cos\left(\frac{\omega\tau}{2}\right)$

$= \hat{I}\frac{A^2}{2\rho_0 c}e^{-2\alpha x}\underbrace{\cos\left(\frac{\omega\tau}{2}\right)}_{\approx 1}$

$\therefore \bar{I} = \hat{I}\frac{A^2}{2\rho_0 c}e^{-2\alpha x} \quad \left(\alpha = \frac{\omega^2\tau}{2c}\right)$

a) $\tau = 0 \Rightarrow I_D = \frac{A^2}{2\rho_0 c} = I_0 = 1\ W/m^2$

b) $\tau = 200ps \Rightarrow I_D = \frac{A^2}{2\rho_0 c}e^{-2\alpha D} = I_0 e^{-2\alpha D} = 0.1\ W/m^2$

Intensity level reduction:

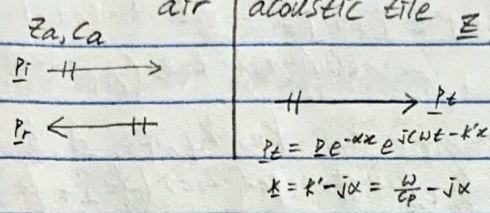
$$IL(x=0) = 10\log_{10}\frac{I_0}{I_{ref}}$$

$$IL(x=D) = 10\log_{10}\frac{I_0 e^{-2\alpha D}}{I_{ref}}$$

$$\Rightarrow \Delta IL = IL(x=D) - IL(x=0) = 10\log_{10} e^{-2\alpha D}$$

$$= 10\log_{10}\frac{1}{10} = -10\,dB$$

⟨Example 2⟩



Given: $z_a = 444\ \frac{Pa\cdot s}{m}$, $c_a = 343\ m/s$

$\underline{Z} = 1000 + j2000\ \frac{Pa\cdot s}{m}$, $C = 600\ m/s$

$\rho = 4\ kg/m^3$

plane wave $f = 1000\ Hz$, $A = 100\ \mu Pa$

Find: a) Speed of sound in the AT, $c_p = ?$

b) Wavelength in the AT, $\lambda = ?$

c) magnitude of the transmitted acoustic pressure in the AT, $|\underline{P}_t(x=0)| = ?$

$$\underline{P}_t = \underline{D} e^{-\alpha x} e^{j(\omega t - k'x)} \Rightarrow c_p = \frac{\omega}{k'},\quad \underline{k} = \frac{\omega}{c_p} - j\alpha$$

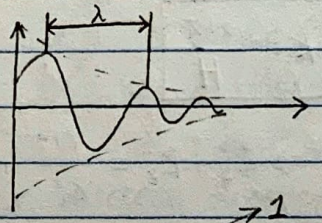
a) $$Re\left\{\frac{\omega}{c}\frac{1}{\sqrt{1+j\omega\tau}}\right\} = \frac{\omega}{c_p} = Re\left\{\frac{\omega}{c}\frac{\rho c}{\underline{Z}}\right\} = Re\left\{\frac{\omega\rho\underline{Z}^*}{|\underline{Z}|^2}\right\}$$

$(\underline{Z} = z_r + jz_i,\ \underline{Z}^* = z_r - jz_i)$

$$= \frac{\omega\rho}{|\underline{Z}|^2} z_r$$

$$\Rightarrow \frac{\omega}{c_p} = \frac{\omega\rho}{|\underline{Z}|^2} z_r \Rightarrow c_p = \frac{|\underline{Z}|^2}{\rho z_r} = 1250\ m/s$$

(b) $\lambda = \frac{c_p}{f}$



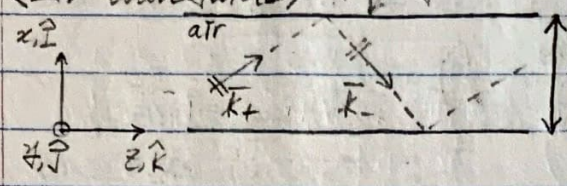
$$\lambda = \frac{c_p}{f} = 1.25\ m$$

(c) $$|\underline{P}_t(x=0)| = |\underline{A}||\underline{T}|\underbrace{e^{-\alpha\cdot 0}}_{\to 1} = 100\ \mu Pa$$

$$\underline{T} = \frac{2\underline{Z}}{\underline{Z} + \underline{Z}_a} = \frac{2|1000 + j2000|}{|1444 + j2000|}$$

Lecture 18

<2D Waveguide>



$$\bar{k}_+ = k_x\hat{I} + k_y\hat{J} + k_z\hat{K}$$
$$\bar{k}_- = -k_x\hat{I} + k_y\hat{J} + k_z\hat{K}$$

$$\underline{P}_+ = \underline{A}\, e^{j(\omega t - k_x x - k_y y - k_z z)} \;;\; \underline{P}_- = \underline{B}\, e^{j(\omega t + k_x x - k_y y - k_z z)}$$

$\Rightarrow$ mode ac. (acoustic) pressure : $\underline{P} = \underline{P}_+ + \underline{P}_-$

$$= e^{j(\omega t - k_y y - k_z z)}\left(\underline{A}e^{-jk_x x} + \underline{B}e^{jk_x x}\right)$$

What are the constraints on $k_x, k_y, k_z, \underline{A}, \underline{B}$?

$\underline{V} = \frac{\underline{P}_+}{Z_c}\hat{k}_+ + \frac{\underline{P}_-}{Z_c}\hat{k}_-$

$\Rightarrow$ particle velocity: $\underline{V} = \frac{\underline{A}}{Z_c} e^{j(\omega t - k_x x - k_y y - k_z z)}\left(\frac{k_x}{k_0}\hat{I} + \frac{k_y}{k_0}\hat{J} + \frac{k_z}{k_0}\hat{K}\right)$

$$+ \frac{\underline{B}}{Z_c} e^{j(\omega t + k_x x - k_y y - k_z z)}\left(-\frac{k_x}{k_0}\hat{I} + \frac{k_y}{k_0}\hat{J} + \frac{k_z}{k_0}\hat{K}\right)$$

B.C.s : @ $x = 0$

$$\underline{V}_x = 0 = \frac{k_x}{k_0 Z_0} e^{j(\omega t - k_y y - k_z z)}\left(\underline{A}e^{-jk_x x} - \underline{B}e^{jk_x x}\right)$$

$\Rightarrow \boxed{\underline{A} = \underline{B}}$

@ $x = H$

$$\underline{V}_x = 0 = \frac{k_x}{k_0 Z_0}\underline{A}\, e^{j(\omega t - k_y y - k_z z)}\underbrace{\left(e^{-jk_x H} - e^{jk_x H}\right)}_{-2j\sin(k_x H)}$$

$\Rightarrow \sin(k_x H) = 0 \Rightarrow k_x H = m\pi$ (m: integer)

$\Rightarrow \boxed{k_x = \frac{m\pi}{H}}$

$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

$$\Rightarrow k_y^2 + k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{H}\right)^2 = k_0^2\left[1 - \left(m\frac{\lambda}{2H}\right)^2\right]$$

(Should be larger than 0 to propagate)

$$k_0^2\left[1 - \left(m\frac{\lambda}{2H}\right)^2\right] \geq 0 \Leftrightarrow 1 \geq m\frac{\lambda}{2H} \Leftrightarrow \boxed{\lambda \leq \frac{2H}{m}}$$

$$\Leftrightarrow \frac{c}{f} \leq \frac{2H}{m} \Leftrightarrow \boxed{f \geq \frac{mc}{2H}}$$

$\underline{P}$ in terms of m ($\underline{P}_m$)

$$\underline{P} = \underline{A}\, e^{j(\omega t - k_y y - k_z z)}\underbrace{\left(e^{-jk_x x} + e^{jk_x x}\right)}_{2\cos(k_x x)}$$

$$= 2\underline{A}\cos\left(\frac{m\pi}{H}x\right) e^{j(\omega t - k_y y - k_z z)} \quad \text{where } k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{H}\right)^2$$

$\underline{P}_m$: $\underline{P}$ for mode m

$$\Rightarrow \underline{P}_m = \underline{A}_m \cos\left(\frac{m\pi}{H}x\right) e^{j(\omega t - k_y y - k_z z)}$$

Notes: ① Mode 0 always propagates in this example because $f > 0$

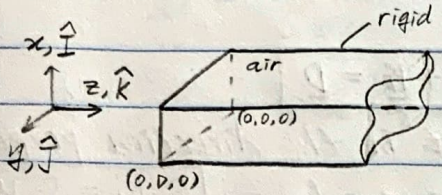
② Mode m propagates when

$$f \geq f_m = \frac{mc}{2H} \quad ; \quad \lambda \leq \lambda_m = \frac{2H}{m}$$

└ threshold frequency └ threshold wavelength

③ For a give frequency f, exactly $1 + \left\lfloor \frac{2fH}{c} \right\rfloor$ modes propagate

⟨Rectangular waveguide⟩



$$\underline{P}_{m,n} = \underline{A}_{m,n} \cos\left(\frac{m\pi}{H}x\right) \cos\left(\frac{n\pi}{D}y\right) e^{j(\omega t - k_z z)}$$

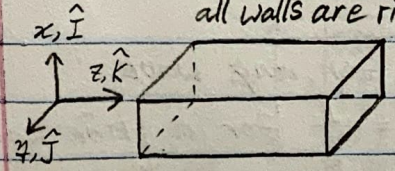
Where $k_z^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{H}\right)^2 - \left(\frac{n\pi}{D}\right)^2 \geq 0$

Notes: ① First mode (fundamental) always propagates

$$\underline{P}_{0,0} = \underline{A}_{0,0} e^{j(\omega t - k_0 z)}$$

② Mode m propagates when $f \geq f_{m,n} = \frac{c}{2}\sqrt{\left(\frac{m}{H}\right)^2 + \left(\frac{n}{D}\right)^2}$

⟨Rectangular cavity⟩



$$\underline{P}_{m,n,r} = \underline{A}_{m,n,r} \cos\left(\overbrace{\frac{m\pi}{H}x}^{k_x}\right) \cos\left(\overbrace{\frac{n\pi}{D}y}^{k_y}\right) \cos\left(\overbrace{\frac{r\pi}{L}z}^{k_z}\right) e^{j\omega t}$$

Where $\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{D}\right)^2 + \left(\frac{r\pi}{L}\right)^2 = \left(\frac{\omega}{c}\right)^2$

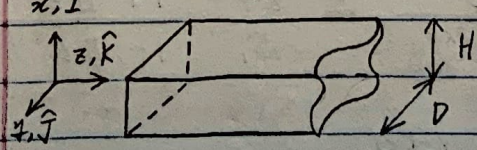
Notes: ① Only discrete frequencies can exist (natural frequencies)

$$f_{m,n,r} = \frac{c}{2}\sqrt{\left(\frac{m}{H}\right)^2 + \left(\frac{n}{D}\right)^2 + \left(\frac{r}{L}\right)^2}$$

⇒ Mode 0 does not propagate

② Any mode (m,n,r) is a superposition of 2, 4, or 6 plane waves

⟨Example⟩



Given: $H = 2\text{cm}$, $D = 4\text{cm}$

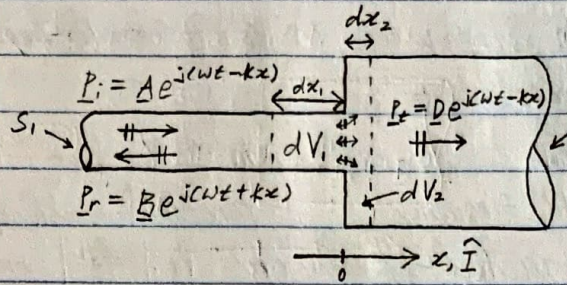
Find: cutoff frequency for which at most 2 modes propagate

↙ larger denominator would give smaller f_m

2nd mode: (0,1) or ~~(1,0)~~ $\Rightarrow f_{0,1} = \frac{343}{2} \frac{1}{4\times10^{-2}} = 4287.5$ Hz

3rd mode: (0,2) or (1,0) $\Rightarrow f_{0,2} = f_{1,0} = 8575$ Hz

Lecture 19



Given: S_1, S_2, c, Z_c

Find: $\underline{R} = \frac{P_r}{P_i}\Big|_{@\text{interface}} = \frac{B}{A} = ?$

$I = \frac{P_t}{P_i}\Big|_{@\text{interface}} = \frac{D}{A} = ?$

Volumetric velocity: $\frac{dV}{dt}\hat{k} = \frac{S dx}{dt}\hat{k} = S\bar{v}$ ← particle velocity

$\Rightarrow$ Pipe 1: $\frac{dV_1}{dt}\hat{I} = S_1\bar{v}_1$, Pipe 2: $\frac{dV_2}{dt}\hat{I} = S_2\bar{v}_2$

B.C.s @ $x=0$

① $\underline{P}_i + \underline{P}_r = \underline{P}_t\Big|_{@x=0} \Rightarrow \boxed{\underline{A} + \underline{B} = \underline{D}}$

② Volumetric velocity component in the direction perpendicular to the interface is conserved. $\Rightarrow \boxed{S_1\underline{v}_1 = S_2\underline{v}_2}$ @ $x=0$

(① $\underline{A} + \underline{B} = \underline{D} \Rightarrow 1 + \underline{R} = \underline{T}$

② $S_1\underline{v}_1 = S_2\underline{v}_2\Big|_{@x=0} \Rightarrow S_1\left(\frac{P_i}{Z_c} - \frac{P_r}{Z_c}\right) = S_2\frac{P_t}{Z_c}\Big|_{@x=0}$

$\Rightarrow \frac{S_1}{Z_c}(\underline{A} - \underline{B}) = \frac{S_2}{Z_c}\underline{D} \Rightarrow \frac{S_1}{Z_c}(1 - \underline{R}) = \frac{S_2}{Z_c}\underline{T}$

$\rightsquigarrow \underline{R} = \frac{Z_c/S_2 - Z_c/S_1}{Z_c/S_2 + Z_c/S_1}$, $\underline{T} = \frac{2Z_c/S_2}{Z_c/S_2 + Z_c/S_1}$

(cf) ① Specific impedance: $\underline{Z}_S = \frac{\underline{P}}{\underline{v}}$ for any wave

② Characteristic impedance: $\underline{Z}_c = \frac{\underline{P}}{\underline{v}}$ for a plane wave

③ Acoustic impedance: $\underline{Z} = \frac{\underline{P}}{\underline{v}S} = \frac{\underline{Z}_S}{S}$ for any wave

Note: $\underline{Z}$ may vary with frequency and position

Not always

Calculate Acoustic Impedance

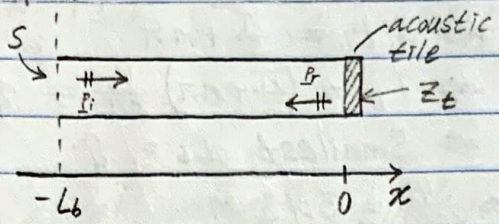
$$\underline{Z}_1(x) = \frac{\underline{P}_i + \underline{P}_r}{(\underline{v}_i + \underline{v}_r)S_1} = \frac{\underline{A}e^{j(\omega t - kx)} + \underline{A}\underline{R}e^{j(\omega t + kx)}}{\left(\frac{\underline{A}}{Z_c}e^{j(\omega t - kx)} - \frac{\underline{A}\underline{R}}{Z_c}e^{j(\omega t + kx)}\right)S_1}$$

$$= \frac{Z_c}{S_1}\frac{e^{-jkx} + \underline{R}e^{jkx}}{e^{-jkx} - \underline{R}e^{jkx}} = \frac{Z_c}{S_1}\frac{1 + \underline{R}e^{2jkx}}{1 - \underline{R}e^{2jkx}} \quad (k = \frac{\omega}{c})$$

$$\underline{Z}_2(x) = \frac{\underline{P}_t}{\underline{v}_t S_2} = \frac{\underline{D}e^{j(\omega t - kx)}}{\frac{\underline{D}}{Z_c}e^{j(\omega t - kx)}S_2} = \frac{Z_c}{S_2}$$

Lecture 20

⟨Example 1⟩



Given: $\underline{P_i} = \underline{A}\, e^{j(\omega t - kx)}$, $\omega = 2\pi(100\,Hz)$

$Z_c = 444\ \frac{Pa\cdot s}{m}$, $L_b = 1\,m$,

$\underline{Z_t} = 10^6 + j10^5\ \frac{Pa\cdot s}{m^3}$, $S = 1\,cm^2$

Find: $\underline{Z}(x=-L_b) = ?$

$\underline{P_i} = \underline{A}\, e^{j(\omega t - kx)}$, $\underline{P_r} = \underline{A}\,\underline{R}\, e^{j(\omega t + kx)}$

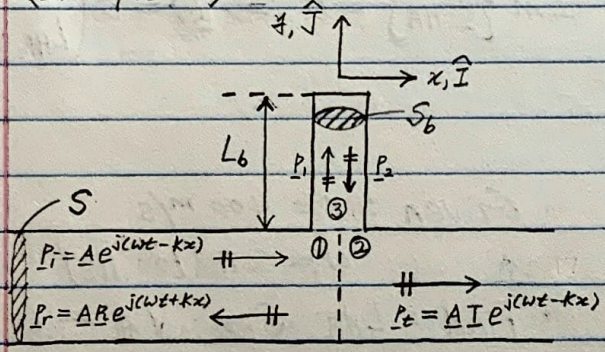
$$\Rightarrow \underline{Z_t} \triangleq \frac{\underline{P_t}}{S\,\underline{V_t}} \left(\underline{P_t}: \text{total pressure inside tile}\right) = \frac{\underline{P_i} + \underline{P_r}}{S(\underline{V_i} + \underline{V_r})}\Bigg|_{@x=0}$$

$$= \underline{Z}(x=0) = \frac{Z_c}{S}\frac{1+\underline{R}}{1-\underline{R}} \Rightarrow \underline{R} = \frac{\underline{Z_t} - Z_c/S}{\underline{Z_t} + Z_c/S} = -0.63 + j0.03$$

$$\Rightarrow \underline{Z}(x=-L_b) = \frac{Z_c}{S} = \frac{1+\underline{R}e^{2jk(-L_b)}}{1-\underline{R}e^{2jk(-L_b)}} = 9.69 + j9.34\ \frac{MPa\cdot s}{m^3}$$

Note: $\underline{Z}(x=0) = \underline{Z_t}$

⟨Example 2⟩



Given: $Z_c = 444\ \frac{Pa\cdot s}{m}$, $c = 343\ m/s$,

$f = 100\ Hz$, $S = \pi\left(\frac{5cm}{2}\right)^2$

Find: $L_b = ?$, $S_b = ?$ that makes $\underline{T} = 0$

B.C.s

① Pressure: $\underline{P_i} + \underline{P_r} = \underline{P_t} = \underline{P_1} + \underline{P_2} \triangleq \underline{P_b}$

② Volumetric particle velocity: $S(\underline{V_i} + \underline{V_r}) = S\,\underline{V_t} + S_b\,\underline{V_b}$

① $\underline{A}e^{j\omega t} + \underline{A}\underline{R}e^{j\omega t} = \underline{A}\underline{T}e^{j\omega t} = \underline{P_b} \Rightarrow 1 + \underline{R} = \underline{T} = \frac{\underline{P_b}}{\underline{A}e^{j\omega t}}$

② $S\left(\frac{\underline{A}e^{j\omega t}}{Z_c} - \frac{\underline{A}\underline{R}e^{j\omega t}}{Z_c}\right) = S\frac{\underline{A}\underline{T}e^{j\omega t}}{Z_c} + S_b\,\underline{V_b}$

$$\Rightarrow 1 - \underline{R} = \underline{T} + \frac{(Z_c/S)\,S_b\,\underline{V_b}}{\underline{A}e^{j\omega t}} = \underline{T} + \frac{Z_0 S_b \underline{V_b}}{\underline{A}e^{j\omega t}} \quad \left(Z_0 = \frac{Z_c}{S}\right)$$

R': branch에서의 R

③ From ⟨Example 1⟩:

$$\frac{\underline{P_b}}{S_b\underline{V_b}} = \frac{Z_c}{S_b}\frac{1+\underline{R}'e^{2jk(-L_b)}}{1-\underline{R}'e^{2jk(-L_b)}} = \frac{Z_c}{S_b}\frac{1+e^{2jk(-L_b)}}{1-e^{2jk(-L_b)}} \quad (R'=1 \text{ because rigid wall})$$

$$= \frac{Z_c}{S_b}\frac{e^{jkL_b}+e^{-jkL_b}}{e^{jkL_b}-e^{-jkL_b}} = \frac{Z_c}{S_b}\frac{2\cos(kL_b)}{2j\sin(kL_b)} = \frac{Z_c}{S_b}(-j)\cot an(kL_b)$$

① $\underline{T} = 1 + \underline{R} = \frac{\underline{P_b}}{\underline{A}e^{j\omega t}}$

② $1 - \underline{R} - \underline{T} = \frac{Z_0 S_b \underline{V_b}}{\underline{A}e^{j\omega t}}$

$$\frac{\underline{T}}{1-\underline{R}-\underline{T}} = \frac{\underline{T}}{2-2\underline{T}} = \frac{1}{2}\,\frac{\underline{P_b}}{Z_0\,S_b\,\underline{V_b}} \overset{③}{=} \frac{\underline{Z_b}}{Z_0}$$

$$\Rightarrow \underline{T} = \frac{1}{1 + \frac{Z_0}{2\underline{Z_b}}} \Rightarrow \underline{T} = 0 \text{ when } \underline{Z_b} = 0$$

Lecture 21

Continued from previous page

Find how to satisfy $\underline{Z}_b = \frac{Z_0}{S_b}(-j)\cot(kL_b) = 0$

$\Rightarrow \tan(kL_b) = \infty \Rightarrow kL_b = \frac{\pi}{2} + n\pi$

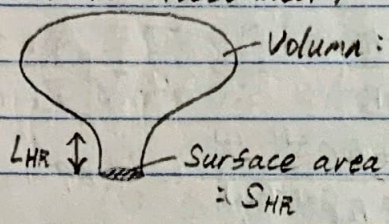
$\Rightarrow L_b = \left(\frac{\pi}{2} + n\pi\right)\frac{\lambda}{2\pi} = \frac{\lambda}{4} + \frac{n\lambda}{2}$

$\Rightarrow$ Smallest $L_b = \frac{\lambda}{4}$

air: $c = 343\ m/s \Rightarrow \lambda = \frac{343}{100} = 3.43\ m$

$\Rightarrow L_b = \frac{\lambda}{4} = 25.75\ cm$

⟨Helmholz Resonator⟩



$V_{HR} \ll \lambda^3$

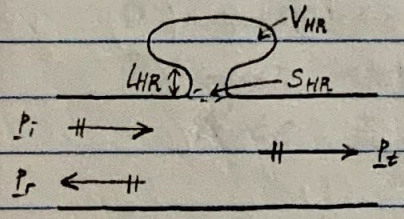
$S_{HR} \ll \lambda^2$

$L_{HR} \ll \lambda$

Impedance at the enterance $\underline{Z}_{HR} = \rho_0 c \frac{k^2}{2\pi} + j\left(\frac{\rho_0 L_{HR}}{S_{HR}}\omega - \frac{\rho_0 c^2}{V_{HR}\omega}\right)$

Resonates when $Im\{\underline{Z}_{HR}\} = 0 \Rightarrow \omega = c\left(\frac{S_{HR}}{L_{HR} V_{HR}}\right)^{\frac{1}{2}}$

⟨Example⟩



Given : $c = 400\ m/s$

$\omega = 2\pi(100\ Hz)$

Find : $L_{HR}, S_{HR}, V_{HR} = ?$ that makes $\underline{T} = 0$

$$\underline{T} = \frac{1}{1 + \frac{Z_0}{2\underline{Z}_{HR}}}$$

$$\Rightarrow |\underline{T}| = \frac{1}{\left|1 + \frac{Z_0}{2\underline{Z}_{HR}}\right|} = \frac{1}{\left|1 + \frac{Z_0}{2Z_R + 2jZ_I}\right|} = \frac{1}{\left|1 + \frac{2Z_0(Z_R - jZ_I)}{4(Z_R^2 + Z_I^2)}\right|}$$

$$= \frac{1}{\left|1 + \frac{Z_0 Z_R}{2(Z_R^2 + Z_I^2)} - j\frac{Z_0 Z_I}{2(Z_R^2 + Z_I^2)}\right|}$$

$$\Rightarrow |\underline{T}|^2 = \frac{1}{\left(1 + \frac{1}{2}\frac{Z_0 Z_R}{Z_R^2 + Z_I^2}\right)^2 + \left(\frac{1}{2}\frac{Z_0 Z_I}{Z_R^2 + Z_I^2}\right)^2} = \frac{1}{1 + \frac{Z_0 Z_R}{Z_R^2 + Z_I^2} + \frac{1}{4}\frac{Z_0^2}{Z_R^2 + Z_I^2}}$$

$$= \frac{4(Z_R^2 + Z_I^2)}{4(Z_R^2 + Z_I^2) + 4Z_0 Z_R + Z_0^2} = \frac{4}{4 + \frac{4Z_0 Z_R + Z_0^2}{Z_R^2 + Z_I^2}}$$

$$\min |\underline{I}|^2 = \frac{4}{4 + \frac{4 Z_0 Z_R + Z_0^2}{Z_R^2}} \quad \text{when} \quad Z_I^2 = 0$$

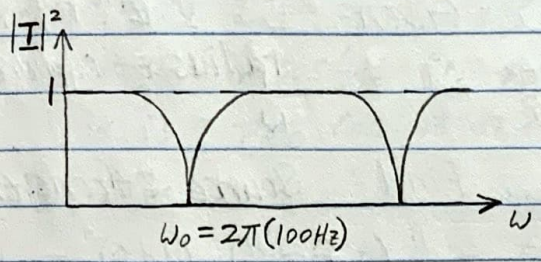
$$\Downarrow$$

$$\left(\frac{\omega}{c}\right)^2 = \frac{S_{HR}}{L_{HR} V_{HR}}$$

Typical values: $L_{HR} = 10\,mm$, $S_{HR} = 10^{-3}\,m^2$,

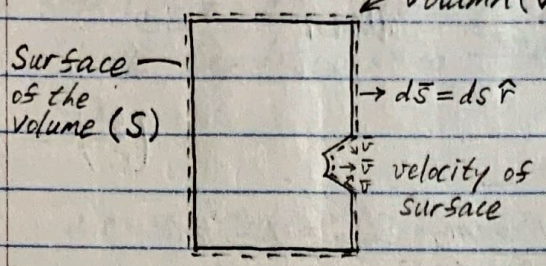
$V_{HR} = \frac{S_{HR}}{L_{HR}} \left(\frac{c}{\omega}\right)^2 = 4.05$ liters

(5) $Z_R = 202 \frac{Pa \cdot s}{m^3}$, $Z_0 = 516{,}000 \frac{Pa \cdot s}{m^3}$



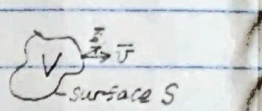
Lecture 22

Source Strength

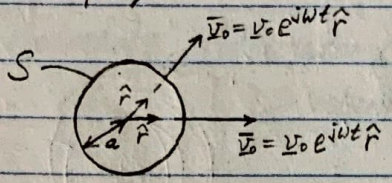


Source strength (q)
: rate of change of the volume encompassing the source

$$q(t) \triangleq \frac{dV}{dt} = \int_S \bar{v} \cdot d\bar{s}$$



〈Example〉



Given: $\underline{\bar{v}}_0 = \underline{v}_0 e^{j\omega t}\hat{r}$, radius of sphere a, ω

Find: Source strength $q(t) = ?$

$$q(t) = \int_S \underline{\bar{v}}_0 \cdot d\bar{s} = \int_S (\underline{v}_0 \hat{r} e^{j\omega t}) \cdot (ds\,\hat{r}) = \int_S \underline{v}_0 e^{j\omega t} ds$$

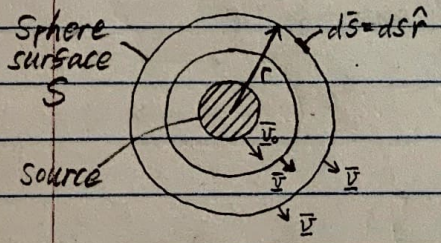
$$= \underline{v}_0 e^{j\omega t} \int_S ds = \underline{v}_0 e^{j\omega t} 4\pi a^2 = \underline{4\pi a^2 v_0} e^{j\omega t}$$

Complex Source Strength ($\underline{Q}$)

$\underline{Q}$ = Complex source strength

$$\Rightarrow q(t) = \underline{Q} e^{j\omega t} = 4\pi a^2 \underline{v}_0 e^{j\omega t}$$

〈Spherical Wave〉



$$\underline{p} = \frac{A}{r} e^{j(\omega t - kr)}$$

unknown

Scalars (Not vectors!)

A is the only unknown in this equation

$\Rightarrow$ Strategy: $\underline{p} \xrightarrow{①} \underline{\bar{v}} \xrightarrow{②} \underline{\bar{v}}\,(r=a)$

$$\underline{\bar{v}}_0 e^{j\omega t} = \frac{Q}{4\pi a^2} e^{j\omega t}\hat{r}$$

Conservation of momentum: $③\ \nabla \underline{p} + \rho_0 \frac{\partial \underline{\bar{v}}}{\partial t} = 0$

Derivation provided in the lecture note →

$$\underline{\bar{v}} = \frac{\underline{p}}{Z_c}\left(1 + \frac{1}{jkr}\right)\hat{r} \quad \cdots ①$$

$$\underline{Z}_s = \frac{\underline{p}}{\underline{v}} = Z_c \frac{1}{1 + \frac{1}{jkr}}$$

Note: ① $\underline{p}$ and $\underline{v}$ are not in phase

② When $r \gg \lambda$ (Very far away), $\underline{p} = \frac{A}{r} e^{j(\omega t - kr)}$ ($\frac{A}{r}$ becomes almost const.)

$$\underline{\bar{v}} \simeq \frac{\underline{p}}{Z_c}\hat{r}$$

$$\underline{\vec{v}}(r=a) = \frac{\underline{p}}{z_c}\left(1+\frac{1}{jka}\right)\hat{r} = \frac{\frac{\underline{A}}{a}e^{j(\omega t-ka)}}{\rho_0 c}\left(1+\frac{1}{jka}\right)\hat{r}$$

$$= \frac{\underline{Q}}{4\pi a^2}e^{j\omega t}\hat{r}$$

$$\Rightarrow \boxed{\underline{A} = \frac{\underline{Q}\rho_0 c}{4\pi a}\frac{e^{jka}}{1+\frac{1}{jka}}} \quad \text{Amplitude}$$

Intensity: $\bar{I} \triangleq \frac{1}{T}\int_0^T Re\{\underline{p}\}\, Re\{\underline{\vec{v}}\}\, dt$

$$\left(Re\{\underline{p}\} = \frac{A}{r}\cos(\omega t - kr + \phi_A)\right.$$

$$\left.Re\{\underline{\vec{v}}\} = \frac{A}{r\rho_0 c}\cos(\omega t - kr + \phi_A + \beta)\sqrt{1+\left(\frac{1}{kr}\right)^2}\,\hat{r}\right.$$

$$1+\frac{1}{jkr} = \sqrt{1+\left(\frac{1}{kr}\right)^2}\,e^{j\beta}$$

$$\beta = -\operatorname{atan}\frac{1}{kr}$$

$$\Rightarrow \bar{I} = \frac{|A|^2}{r^2\rho_0 c}\sqrt{1+\left(\frac{1}{kr}\right)^2}\,\frac{1}{T}\int_0^T \cos(\omega t-kr+\phi_A)\cos(\omega t-kr+\phi_A+\beta)\,dt\,\hat{r}$$

$$\cos a\cos b = \frac{\cos(a+b)+\cos(a-b)}{2}$$

$$= \int_0^T \frac{\cos(2\omega t-2kr+2\phi_A+\beta)+\cos\beta}{2}\,dt$$

$$= \frac{T}{2}\cos\beta$$

$$= \frac{|A|^2}{2r^2\rho_0 c}\sqrt{1+\left(\frac{1}{kr}\right)^2}\cos\left(\operatorname{atan}\frac{1}{kr}\right)\hat{r}$$

$$\cos(\operatorname{atan} x) = \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{|A|^2}{2r^2\rho_0 c}\sqrt{1+\left(\frac{1}{kr}\right)^2}\,\frac{1}{\sqrt{1+\left(\frac{1}{kr}\right)^2}}\,\hat{r}$$

$$\Rightarrow \boxed{\bar{I} = \frac{|A|^2}{2r^2\rho_0 c}\hat{r}} \quad \text{Intensity}$$

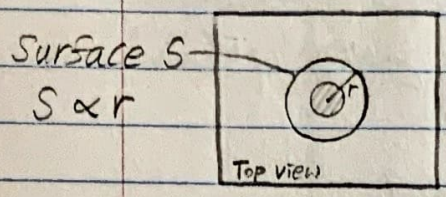
Power $P \triangleq \int_S \bar{I}\cdot d\bar{s} = \int_S \left(\frac{|A|^2}{2\rho_0 c}\frac{1}{r^2}\hat{r}\right)\cdot(ds\,\hat{r})$

$$= \frac{|A|^2}{2\rho_0 c}\frac{1}{r^2}\int_S ds = \frac{|A|^2}{2\rho_0 c}\frac{4\pi r^2}{r^2}$$

$$\Rightarrow \boxed{P = \frac{2\pi|A|^2}{\rho_0 c}} \quad \text{Power}$$

⟨Cylindrical Wave⟩

$$\underline{P} = \frac{A}{\sqrt{r}} e^{j(\omega t - kr)}$$



⟨Monopole Sources⟩ $\underline{P} = \frac{A}{r} e^{j(\omega t - kz)}$

A monopole source (Simple Source) is a spherical source whose $a \ll \lambda \Leftrightarrow ka = \frac{2\pi a}{\lambda} \to 0$

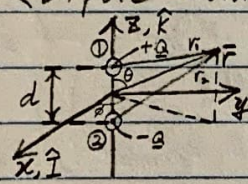
$$\underline{A} = \frac{Q\rho_0 c}{4\pi a} \frac{e^{jka}}{1 + \frac{1}{jka}} \xrightarrow{a \ll \lambda} \frac{Q\rho_0 c}{4\pi a} \frac{1}{\frac{1}{jka}} = \frac{Q\rho_0 c}{4\pi}(jk)$$

$$\Rightarrow \boxed{\underline{A} = \frac{Q\rho_0 c}{4\pi}(jk)}$$ Amplitude of a Monopole Source

Note: ① The size of the monopole (a) does not influence the spherical wave launched by the source.

★ ② The power launched by the monopde increases with frequency.

⟨Dipole Sources⟩



$d \ll \lambda$, ① and ② have the opposite phase

We are interested in $\underline{P}$ for $|\bar{r}| \gg \lambda$

$$\underline{P} = \underline{P}_1 + \underline{P}_2 = \frac{A}{|\bar{r}_1|} e^{j(\omega t - k|\bar{r}_1|)} + \frac{-A}{|\bar{r}_2|} e^{j(\omega t - k|\bar{r}_2|)} \quad (\text{Where } \underline{A} = \frac{Q\rho_0 c}{4\pi} jk)$$

($-A = A e^{j\pi}$)

① If $\bar{r}$ is in the (x, y) plane $\Rightarrow \underline{P} = 0$

② If $\bar{r}$ is on the z axis: $\bar{r} = z\hat{k}$

$$\underline{P} = \frac{A}{r - d/2} e^{j(\omega t - kr + k\frac{d}{2})} - \frac{A}{r + d/2} e^{j(\omega t - kr - k\frac{d}{2})}$$

$$= \frac{A}{r - d/2} e^{j(\omega t - kr)} \left[e^{jk\frac{d}{2}} - \frac{r - d/2}{r + d/2} e^{-jk\frac{d}{2}} \right]$$

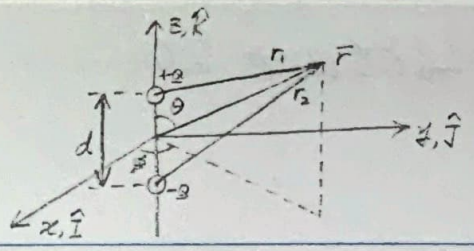
$$\cong \frac{A}{r} e^{j(\omega t - kr)} 2j \sin\left(\frac{kd}{2}\right)$$

$$\cong \frac{A}{r} jkd\, e^{j(\omega t - kr)}$$

↓ $r - \frac{d}{2} \cong r + \frac{d}{2} \cong r$

↓ $\sin(x) \approx x$ when $x \approx 0$

Lecture 23



③ If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\underline{I} = \frac{A}{r_1} e^{j(\omega t - kr_1)} \left[1 - \frac{r_1}{r_2} e^{-jk(r_2 - r_1)} \right]$$

$$\cong \frac{A}{r} e^{j(\omega t - kr)} \left[1 - e^{-jk(r_2 - r_1)} \right]$$

$$= \frac{A}{r} e^{j(\omega t - kr)} \, 2j \, e^{-jk\frac{r_2 - r_1}{2}} \sin\left[k\left(\frac{r_2 - r_1}{2}\right) \right]$$

$$\cong 2jk \frac{A}{r} e^{j(\omega t - kr)} e^{-jk\frac{r_2 - r_1}{2}} \frac{r_2 - r_1}{2}$$

$1 - e^{j\alpha} = e^{j\frac{\alpha}{2}}\left(e^{-j\frac{\alpha}{2}} - e^{j\frac{\alpha}{2}}\right) = e^{j\frac{\alpha}{2}}(-2j)\sin\left(\frac{\alpha}{2}\right)$

$\sin(x) \approx x$ when $x \approx 0$

Very widely used approximations

$$r_1 = |\bar{r}_1| = \left|\bar{r} - \frac{d}{2}\hat{k}\right| = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$r_2 = |\bar{r}_2| = \left|\bar{r} + \frac{d}{2}\hat{k}\right| = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

$$\Rightarrow \sqrt{x^2 + y^2 + \left(z \pm \frac{d}{2}\right)^2} = \sqrt{x^2 + y^2 + z^2 + \left(\frac{d}{2}\right)^2 \pm zd}$$

$$= \sqrt{r^2 + \left(\frac{d}{2}\right)^2 \pm zd} = r\sqrt{1 + \left(\frac{d}{2r}\right)^2 \pm \frac{z}{r}\frac{d}{r}}$$

($\left(\frac{d}{2r}\right)^2 \to 0$, $\frac{z}{r} = \cos\theta$)

$$\cong r\sqrt{1 \pm \frac{d}{r}\cos\theta}$$

$$\cong r\left(1 \pm \frac{d}{2r}\cos\theta\right) = r \pm \frac{d}{2}\cos\theta$$

$\lim_{a \to 0} \sqrt{1 \pm a} = 1 \pm \frac{a}{2}$

$$\Rightarrow r_2 - r_1 \cong \left(r + \frac{d}{2}\cos\theta\right) - \left(r - \frac{d}{2}\cos\theta\right) = d\cos\theta$$

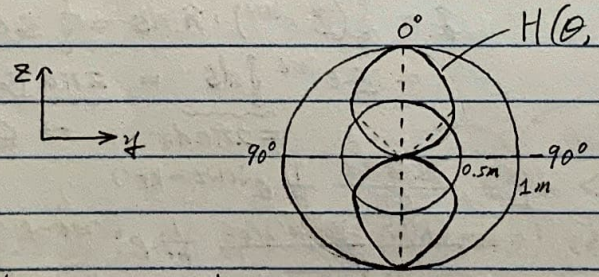
$$\cong 2jk \frac{A}{r} e^{j(\omega t - kr)} e^{-jk\frac{d\cos\theta}{2}} \frac{d\cos\theta}{2}$$

($e^{-jk\frac{d\cos\theta}{2}} \to 1$)

$$\Rightarrow \boxed{\underline{I} = -k^2 d \frac{Q\rho_0 c}{8\pi} \frac{1}{r} \cos\theta \, e^{j(\omega t - kr)}} \quad \times \text{Wrong}$$

$$|\underline{I}| = \underbrace{k^2 d \frac{|Q|\rho_0 c}{8\pi} \frac{1}{r}}_{R(r)} \underbrace{\cos\theta}_{H(\theta,\phi)} = R(r)\,H(\theta,\phi)$$

$H(\theta,\phi)$: direction factor



⟨Directivity⟩

$$D = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} H^2(\theta,\phi)\sin\theta \, d\theta \, d\phi}$$

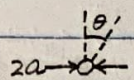
If symmatric for ϕ : $D = \dfrac{2}{\int_0^{\pi} H^2(\theta)\sin\theta \, d\theta}$

For dipole: $D = \dfrac{2}{\int_0^{\pi} \cos^2\theta \sin\theta} = \dfrac{3}{2}$

Directivity Index : $DI = 10\log_{10} D$ [dB]

Lecture 24

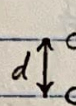
⟨Monopole Source⟩



$$\underline{P} = \frac{jk\underline{Q}\rho_0 c}{4\pi}\frac{1}{r}e^{j(\omega t - kr)}$$

$$\Rightarrow |\underline{P}| = \underbrace{\frac{k|\underline{Q}|\rho_0 c}{4\pi}\frac{1}{r}}_{R(r)}\underbrace{1}_{H(\theta)} \qquad (\text{ES})\ |\underline{P}| \propto \text{ES}$$

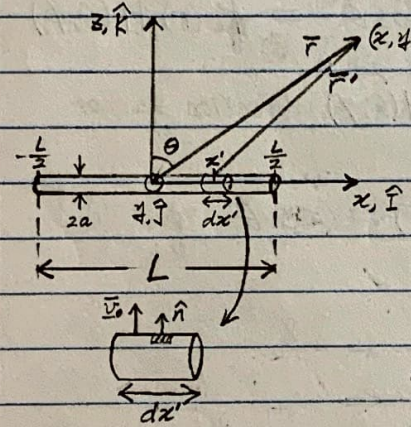
⟨Dipole Source⟩



$$\underline{P} = \frac{k^2 d\underline{Q}\rho_0 c}{4\pi}\frac{1}{r}e^{j(\omega t - kr)}\cos\theta$$

$$\Rightarrow |\underline{P}| = \underbrace{\frac{k^2 d|\underline{Q}|\rho_0 c}{4\pi}\frac{1}{r}}_{R(r)}\underbrace{|\cos\theta|}_{H(\theta)} \qquad (\text{ES})\ |\underline{P}| \propto \text{ES}^2$$

⟨Line Source⟩



Given: Surface pulsates with velocity $\underline{v}_0 e^{j\omega t}$, $a \ll \lambda$

Find: $\underline{P}(\bar{r}, t) = ?$ in far field $|\bar{r}| \gg \lambda$

$$\underline{q} = \int_S (\underline{\bar{v}}_0 e^{j\omega t})\cdot\hat{n}\,ds = \int_S \underline{v}_0 e^{j\omega t}\,ds$$

$$= \underline{v}_0 e^{j\omega t}\underbrace{\int_S ds}_{=2\pi a dx'} = \underbrace{2\pi a \underline{v}_0 dx' e^{j\omega t}}_{\to \underline{Q}}$$

$$\Rightarrow d\underline{P} = \frac{jk\underline{Q}\rho_0 c}{4\pi}\frac{1}{r'}e^{j(\omega t - kr')}$$

$$= \frac{jk(2\pi a \underline{v}_0 dx')\rho_0 c}{4\pi}\frac{1}{r'}e^{j(\omega t - kr')}$$

$$= \frac{jka\underline{v}_0 dx'\rho_0 c}{2}\frac{1}{r'}e^{j(\omega t - kr')}$$

$$\Rightarrow \underline{P} = \int d\underline{P} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{jka\underline{v}_0\rho_0 c}{2}\frac{1}{r'}e^{j(\omega t - kr')}dx'$$

$$= \frac{jka\underline{v}_0\rho_0 c}{2}e^{j\omega t}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{1}{r'}e^{-jkr'}dx'$$

$$\frac{1}{r'} \approx \frac{1}{r} \qquad e^{-jkr'} \approx e^{-jk(r - x'\sin\theta)}$$

Approximation $r' = |\bar{r}'| = \sqrt{(x-x')^2 + y^2 + z^2} = \sqrt{x^2+y^2+z^2+x'^2-2xx'}$

$= \sqrt{r^2 + x'^2 - 2xx'}$

$r^2 \gg x'^2$ ↘

$\sqrt{1 \pm a} \cong 1 \pm \frac{a}{2}$ ↘

$\cong \sqrt{r^2 - 2xx'} = r\sqrt{1 - 2\frac{x}{r}\frac{x'}{r}} = r\sqrt{1 - 2\sin\theta\frac{x'}{r}}$

$\cong r\left(1 - \frac{x'}{r}\sin\theta\right) = r - x'\sin\theta$

$$\Rightarrow \underline{P} \cong \frac{jka\underline{v}_0\rho_0 c}{2}\frac{1}{r}e^{j(\omega t - kr)}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{jkx'\sin\theta}\,dx'$$

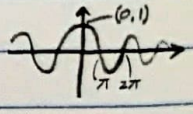
$$= \frac{jka\underline{v}_0\rho_0 c}{2}\frac{1}{r}e^{j(\omega t - kr)}\frac{1}{jk\sin\theta}\left[e^{jkx'\sin\theta}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{jka\underline{v}_0\rho_0 c}{2}\frac{1}{r}e^{j(\omega t - kr)}\frac{1}{jk\sin\theta}\,2j\sin\left(\frac{kL}{2}\sin\theta\right)$$

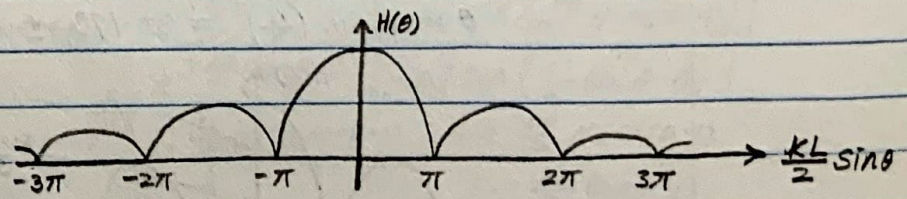
$$= \frac{jka\underline{v}_0\rho_0 c}{2}\frac{1}{r}e^{j(\omega t - kr)}\,L\,\frac{\sin\left(\frac{kL}{2}\sin\theta\right)}{\frac{kL}{2}\sin\theta}$$

$$= \frac{jkaL\underline{v}_0\rho_0 c}{2}\frac{1}{r}e^{j(\omega t - kr)}\,\mathrm{sinc}\left(\frac{kL}{2}\sin\theta\right)$$

$\mathrm{sinc}(x) = \frac{\sin x}{x}$



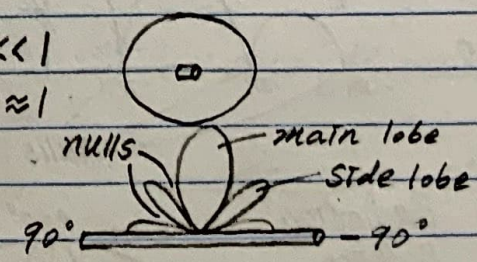
$$\Rightarrow |\underline{P}| = \underbrace{\frac{kaL|\underline{v}_0|\rho_0 c}{2}\frac{1}{r}}_{R(r)}\underbrace{\left|\mathrm{sinc}\left(\frac{kL}{2}\sin\theta\right)\right|}_{H(\theta)}$$



Case 1: $kL \ll 1 \Rightarrow \left(\frac{kL}{2}\sin\theta\right) \ll 1$

$\Rightarrow \mathrm{sinc}\left(\frac{kL}{2}\sin\theta\right) \approx 1$

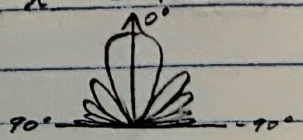
Case 2: $kL \gg 1$

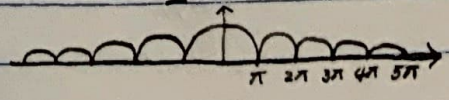


Example: $L = 5\lambda = 7.5\text{cm}$

$f = 100\,\text{kHz} \Rightarrow \lambda = \frac{1500}{100\times10^3} = 1.5\text{cm}$

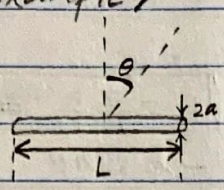
$\Rightarrow \frac{kL}{2} = \pi\frac{L}{\lambda} = 5\pi \Rightarrow -5\pi \leq \frac{kL}{2}\sin\theta \leq 5\pi$





Lecture 25

〈Example〉



Given: $L = 3cm$, $a = 1mm$, $f = 150\,kHz$
in water

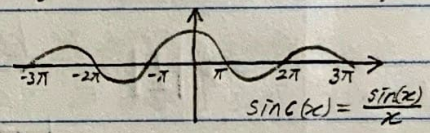
Find: Radiation pattern
(Direction of null & side lobes)

$$\underline{P} = \underbrace{\frac{jkLa\underline{V}_0\rho_0 c}{2}}_{\text{constant term}} \frac{1}{r} e^{j(\omega t - kr)} \operatorname{sinc}\left(\frac{kL}{2}\sin\theta\right)$$

$$|\underline{P}| = \underbrace{\frac{kLa|V_0|\rho_0 c}{2}\frac{1}{r}}_{R(r)} \underbrace{\left|\operatorname{sinc}\left(\frac{kL}{2}\sin\theta\right)\right|}_{H(\theta)}$$

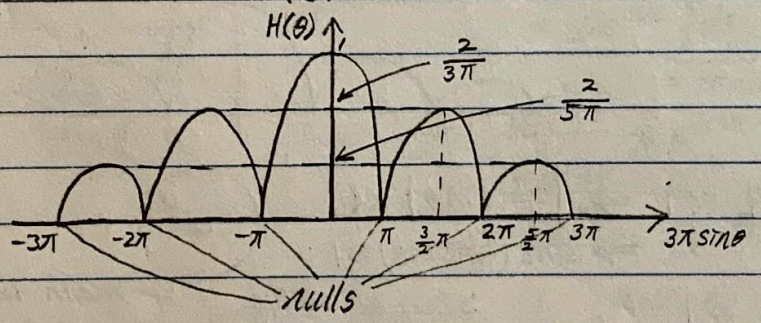
$$H(\theta) = \left|\operatorname{sinc}\left(\frac{2\pi}{10^{-2}}\,\frac{3\times10^{-2}}{2}\sin\theta\right)\right|$$

$$= \left|\operatorname{sinc}(3\pi\sin\theta)\right| \quad (-3\pi \le 3\pi\sin\theta \le 3\pi)$$

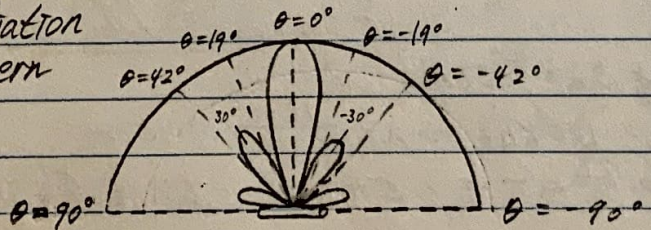


Nulls occur when $3\pi\sin\theta = n\pi$

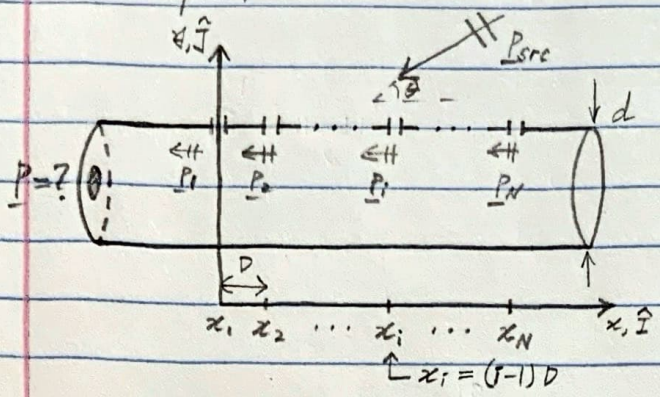
$\Rightarrow \theta = \operatorname{asin}\left(\frac{n}{3}\right) = \pm 19°, \pm 42°$



Radiation Pattern



<Example>



Given: $N = 10$, $D = 5\,mm \ll \lambda$, $d \ll \lambda$ (tube is mono-mode), $\omega = 2\pi \times 1kHz$.

Find: The direction in which the microphone is most sensitive

$$\underline{P}_{src} = \underline{A} e^{j(\omega t - \bar{k}\cdot\bar{r})} = \underline{A} e^{j(\omega t + kx\cos\theta + ky\sin\theta)}$$

$$\bar{k} = k(-\cos\theta\,\hat{I} - \sin\theta\,\hat{J}), \quad \bar{r} = x\hat{I} + y\hat{J} + z\hat{k}$$

$$\underline{P}_i = \underline{B}_i e^{j(\omega t + kx)}$$

$\Rightarrow$ B.C.s : $\underline{P}_i = \underline{P}_{src}$ at $x = (i-1)D$, $y = 0$

$$\underline{B}_i e^{j(\omega t + k(i-1)D)} = \underline{A} e^{j(\omega t + k(i-1)D\cos\theta)}$$

$$\underline{B}_i = \underline{A} e^{jk(i-1)D(\cos\theta - 1)}$$

$$\Rightarrow \underline{P} = \sum_{i=1}^{N} \underline{P}_i = \sum_{i=1}^{N} \underline{A} e^{jk(i-1)D(\cos\theta-1)} e^{j(\omega t + kx)}$$

$$= \underline{A} e^{j(\omega t + kx)} \sum_{i=1}^{N} e^{jk(i-1)D(\cos\theta-1)}$$

$$\sum_{i=1}^{N} a^{i-1} = \frac{1-a^N}{1-a}$$

$$= \underline{A} e^{j(\omega t + kx)} \frac{1 - e^{jkND(\cos\theta-1)}}{1 - e^{jkD(\cos\theta-1)}}$$

$$= \underline{A} e^{j(\omega t + kx)} \frac{e^{j\frac{kD}{2}N(\cos\theta-1)}\left[-2j\sin\left(\frac{kD}{2}N(\cos\theta-1)\right)\right]}{e^{j\frac{kD}{2}(\cos\theta-1)}\left[-2j\sin\left(\frac{kD}{2}(\cos\theta-1)\right)\right]}$$

$$= \underline{A} e^{j(\omega t + kx)} e^{j\frac{kD}{2}(N-1)(\cos\theta-1)} \frac{\sin\frac{kD}{2}N(\cos\theta-1)}{\sin\frac{kD}{2}(\cos\theta-1)}$$

$D \ll \lambda$ & $\sin x \cong x$ for $x \approx 0$

$$\cong \underline{A} e^{j(\omega t + kx)} e^{j\frac{kD}{2}(N-1)(\cos\theta-1)} N\,\mathrm{sinc}\left[\frac{kD}{2}N(\cos\theta-1)\right]$$

$$|\underline{P}| = \underbrace{|A| N}_{\text{Amplitude of incoming sound (|A|) times } N} \underbrace{\left|\mathrm{sinc}\left[\frac{k(DN)}{2}(\cos\theta-1)\right]\right|}_{H(\theta)}$$

(DN: length of tube)

